Referee Report

"Bouncing Geodesics, Singularities, and the Cavity Thermal Product Formula in Asymptotically Flat and de Sitter Black Holes"

Summary

The paper studies how the "bouncing geodesic" phenomenon, familiar from work on asymptotically anti-de Sitter black holes and holographic thermal correlators, extends to asymptotically flat Schwarzschild and Schwarzschild–de Sitter spacetimes. The authors first formulate the relation between null propagation and singularities of retarded Green functions using the local Hadamard form and the propagation of singularities theorem. They then construct radial spacelike geodesics whose large-energy limits approach the curvature singularity and whose proper lengths vanish, and they identify the associated complex Schwarzschild-time separations as bouncing times. The analysis is carried out for asymptotically flat Schwarzschild black holes in general dimension, with special discussion of four and five dimensions, and for five-dimensional Schwarzschild–de Sitter black holes with several choices of anchoring surface. The paper also treats anchoring at null infinity or the cosmological horizon, and it emphasizes that geodesics bouncing off spatial infinity in the de Sitter case do not produce the same Green-function singularities.

The second main part of the paper introduces a reflecting timelike cavity wall and derives a cavity analogue of the thermal product formula. For scalar perturbations, the authors express the two-sided "boundary" correlator at the wall in terms of Wronskians/Jost functions, argue that its inverse is an entire function of order one, and factorize it over the cavity quasinormal frequencies. This leads to the central asymptotic relation

$$t_* \sim \frac{2\pi n}{\omega_n}, \qquad n \to \infty,$$

where t_* is the complex bouncing time and ω_n are high-overtone cavity quasinormal frequencies. The paper supports this relation numerically for scalar perturbations in flat and de Sitter examples and for electromagnetic and gravitational perturbations of a four-dimensional Schwarzschild black hole in a reflecting cavity.

The work addresses an interesting and timely question: whether the singularity-sensitive analytic structures found in AdS correlators have a meaningful counterpart without a holographic boundary. The proposal to use a finite reflecting wall is natural, and the connection between interior null propagation and exterior cavity spectra is potentially valuable. The manuscript is ambitious, technically substantial, and, in my view, likely publishable after revision. However, several central claims are currently stated more strongly than the arguments justify, and some derivational and numerical details should be sharpened before publication.

Recommendation

I recommend major revision. The paper contains enough new and interesting material to merit publication in a high-energy theory journal, but the authors should either strengthen or carefully qualify two core logical steps: the promotion of local Hadamard singularities to the proposed bouncing singularities associated with the curvature-singularity limit, and the proof of the entire-function/product-formula assumptions for the cavity correlator. These issues do not seem fatal, but they affect the status of the main claims and should be addressed explicitly.

Evaluation of Correctness and Logical Structure

The geodesic computations in Sections 3 and 4 appear internally consistent. In the asymptotically flat case, the turning point $r_T = (E^2+1)^{1/(3-D)}$ tends to the curvature singularity as $E \to \infty$, and the proper length tends to zero both in $D = 4$ and in $D > 4$. The resulting finite-radius bouncing times, including $t_*^{\Lambda=0} = 2(\log(1-r_i)+r_i)$ in four dimensions, have the expected imaginary contribution when the analytically continued trajectory crosses the horizons. The discussion of the null-infinity limit is also useful, especially the distinction between the finite part of the time integral and the divergent coordinate-time contribution.

The Schwarzschild–de Sitter analysis is more specialized but convincing at the level presented. The choice of static-sphere time is physically appropriate and avoids a common normalization ambiguity in de Sitter black-hole thermodynamics. The formulas for generic anchoring, for the static-sphere observer, and for the cosmological-horizon limit are clearly organized. The distinction between bounces off the curvature singularity and bounces off spatial infinity is a particularly valuable point: the observation that the latter does not lead to vanishing geodesic length, and hence should not correspond to a retarded Green-function singularity, is conceptually important.

The product-formula derivation is plausible and well motivated. The construction of the retarded Green function from the ingoing horizon solution and the wall solution is standard, and the Wronskian manipulations leading to the two-sided correlator are clear. The extension from Dirichlet to Robin boundary data is also natural for a regular wall. The use of a wall "boundary" correlator to remove insertion-point zeros from the bulk-to-bulk correlator is sensible, though it should be presented as a chosen response prescription rather than as a canonical object.

The weakest part of the logical chain is the proof status of the singularity and product-formula claims. In several places the paper moves from well-established local or conditional statements to global conclusions in a way that needs more qualification. This is especially important because the main advertised relation between bouncing times and cavity QNMs depends on both ingredients.

Novelty and Significance

The paper's main novelty is the extension of the bouncing-geodesic/bouncing-singularity program away from AdS, together with the proposal that a reflecting cavity supplies the right analogue of a holographic boundary for spectral purposes. The connection to high-overtone cavity QNMs is interesting, and the examples involving scalar, electromagnetic, and gravitational perturbations make the proposal more concrete than a purely formal argument would be.

The work is also useful because it separates several concepts that are sometimes conflated: timelike AdS boundary anchoring, null-infinity anchoring, finite-radius cavity anchoring, cosmological-horizon anchoring, and bounces off spatial infinity. The result that spatial-infinity bounces do not produce the relevant Green-function singularities is a good diagnostic of the physical content of the construction.

If the authors clarify the assumptions behind the microlocal and complex-analytic steps, the paper should be of interest to readers working on black-hole interiors, quasinormal modes, thermal correlators, and possible non-AdS analogues of holographic diagnostics.

Major Comments

1. **The propagation-of-singularities argument is stated too strongly for the bouncing problem.**

Section 2 correctly recalls the local Hadamard form and the propagation of singularities theorem for normally hyperbolic operators on smooth Lorentzian manifolds. However, the theorem stated in the manuscript says that if there exists a future-directed null geodesic from x' to x, then the retarded Green function is singular at x. As written, this is stronger than what the preceding argument establishes unless additional hypotheses are imposed. More importantly for this paper, the "bouncing geodesic" is not an ordinary null geodesic through a smooth spacetime region: it is a null limit of spacelike or timelike geodesics that approach a curvature singularity, where the classical manifold itself ends.

The authors should explain precisely how the microlocal theorem is being used in the presence of the black-hole singularity. The propagation theorem propagates wavefront set along null bicharacteristics in the smooth region where the differential operator is defined. It does not by itself justify propagation through, reflection from, or limiting continuation at a curvature singularity. If the intended statement is that the retarded correlator develops a singularity as a limiting consequence of a family of spacelike geodesics whose length tends to zero, then the paper should say so and separate that argument from the standard propagation theorem. If the intended statement is instead based on a known theorem for the retarded fundamental solution on a maximally extended spacetime with the singularity treated as a limiting boundary, the relevant assumptions should be stated explicitly.

This point is central because the paper's physical interpretation rests on the claim that the singularity in the correlator is a direct signature of the curvature singularity. I do not think the claim must be abandoned, but it should be formulated with the correct mathematical caveats.

2. **The proof of the cavity thermal product formula should be made either rigorous under stated assumptions or explicitly conditional.**

   The Wronskian construction and the cancellation of Matsubara poles by the $\sinh(\beta\omega/2)/\omega$ factor are convincing at a formal level. The manuscript then argues that $1/G^{\partial}_{12}(\omega)$ is an entire function of order one. Away from the imaginary axis the scattering-theory input is reasonably clear, but the final step controlling growth along the remaining directions relies on an expected large-imaginary-ω behavior of the correlator, imported by analogy with the AdS complex-WKB analysis and supported by numerical spectra.

   This is an important step, not a minor technicality: the Hadamard factorization and the subsequent spectral-spacing result depend on it. The authors should either provide a self-contained complex-WKB argument for the class of cavity potentials considered here, or state the product formula and the $t_* \sim 2\pi n/\omega_n$ relation as conditional on the relevant exponential-type/growth assumptions. A precise statement such as "assuming the boundary correlator has the large-imaginary-frequency asymptotics controlled by the bouncing saddle" would already make the logical status clearer.

   Relatedly, the paper should specify more carefully the possible singularity structure of the Jost function after continuation into the lower half-plane. The text says that the possible poles occur at Matsubara frequencies and are cancelled in $1/G^{\partial}_{12}$. It would be helpful to spell out why there are no residual branch cuts or higher-order pole complications for the black-hole cavity potentials under consideration, especially for the Robin and gravitational cases.

3. **The de Sitter thermal state and cavity setup need a sharper domain statement.**

   The manuscript says that the Schwarzschild–de Sitter cavity setup is thermal at the black-hole Hawking temperature because the cosmological horizon is cut off by a Dirichlet wall. This is reasonable if the wall is placed inside the static patch, $r_b < r_i < r_c$, and the exterior region containing the cosmological horizon is excluded. However, parts of the geodesic analysis discuss anchoring choices more broadly, including cosmological-horizon anchoring and generic bulk radii. The product-formula section should clearly state the radial domain to which the thermal cavity construction applies, which temperature is used, and how this differs from the separate geodesic computations with horizon or null-boundary anchoring.

This clarification is particularly important because Schwarzschild–de Sitter spacetime does not have a single natural Hartle-Hawking state regular on both horizons when the black-hole and cosmological temperatures differ. The wall setup avoids this issue, but the manuscript should make the avoidance explicit.

4. **The gravitational boundary conditions require more justification.**

   For electromagnetic perturbations, the discussion of perfect electric conductor boundary conditions and vanishing energy flux is clear. For gravitational perturbations, the manuscript states that fixing the induced metric at the wall gives $\delta g_{tt} = 0$, $\delta g_{tA} = 0$, and $\delta g_{AB} = 0$, or equivalently $\tilde{w}(r_i) = 0$, $\tilde{h}_t(r_i) = 0$, and $\tilde{h}_{tt}(r_i) = 0$. It then imposes one Robin condition on the odd Regge–Wheeler master field and one Robin condition on the even Zerilli master field.

   The authors should show explicitly why these master-field conditions are equivalent to the full fixed-induced-metric boundary condition, or state which subset of the induced-metric condition is actually imposed. In the even sector, for example, the text uses $w(r_i) = 0$ to obtain the Zerilli Robin condition, but it is not immediately clear from the presentation whether $h_{tt}(r_i) = 0$ is then automatic by the constraint equations, redundant after gauge fixing, or an additional condition that would overdetermine the QNM problem. Since the gravitational numerical evidence is one of the paper's broader-significance claims, this point deserves a more explicit derivation.

5. **The numerical evidence should be made more reproducible and quantitative.**

   The figures are useful, but the paper should include at least one table or explicit numerical fit showing how $\Delta\omega_n$ approaches $2\pi/t_*$. The comparison is between complex quantities, so the authors should state precisely which branch and ordering convention is used for the high-overtone frequencies, and whether the plotted convergence is for the full complex ratio $\Delta\omega_n t_*/(2\pi)$, its real part, its imaginary part, or its magnitude.

   The Frobenius truncation criterion is described, but more detail would improve confidence: the authors should state the overtone ranges used in the plots, the maximum truncation order for each figure, the root-finding method, and how spurious roots are excluded. For the Schwarzschild–de Sitter examples, it would also be useful to state explicitly that the wall lies within the convergence disk of the horizon expansion, or otherwise explain how convergence is maintained.

Minor Comments

1. The manuscript contains several visible encoding artifacts in the source, for example around en dashes and quotation marks. The authors should check the compiled PDF carefully for mojibake such as "Schwarzschild–de Sitter" appearing incorrectly.

2. The phrase "asymptotically flat ($\Lambda = 0$) and asymptotically de Sitter ($\Lambda > 0$) black holes" appears with a repeated "black holes" in the introduction. There are a few similar small grammatical slips.

3. In Section 2, the global theorem should be stated in terms of the wavefront set of the fundamental solution, or with sufficient hypotheses to make the singular-support inclusion

valid. As currently phrased, it may invite objections from readers familiar with caustics, conjugate points, and possible cancellations.

4. The sign convention for the imaginary part of t_* is explained in words, but it would be helpful to collect the convention once and use it consistently in the flat, de Sitter, and cavity sections.

5. The discussion of the "boundary" correlator at a finite wall is reasonable, but the quotation marks around boundary appear frequently. The authors might define the term once and then use it without repeated scare quotes.

6. The relation $t_{**}^{\Lambda=0}(D+2) = t_*^{\Lambda<0}(D)$ is intriguing. Since the paper mentions possible physical significance, a sentence clarifying whether this is merely an algebraic coincidence of the formulas or tied to a known dimensional-shift relation would help calibrate the claim.

7. The final paragraph has an observational tone. This is acceptable as motivation, but the authors should keep the distinction between an ideal reflecting cavity and realistic astrophysical boundary conditions clear, especially if the related follow-up work is not yet available.

Conclusion

This is a strong and interesting manuscript with a clear path to publication. Its main results are novel and potentially important, and the calculations presented are substantial. Before publication, however, the authors should sharpen the mathematical status of the two central mechanisms: propagation of singularities in relation to the curvature-singularity bounce, and the growth/entire-function assumptions behind the cavity thermal product formula. They should also make the gravitational boundary conditions and numerical comparisons more explicit. With these revisions, I would support publication.