

Referee Report

Lectures on Semiclassical Methods for Composite Operators

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Recommendation: publish after minor revision.

Summary

This manuscript is a long set of lecture notes on semiclassical methods for heavy composite operators in conformal field theory, with the operator ϕ^n and its $O(N)$ generalizations as the principal examples. The notes begin with a self-contained review of conformal symmetry, primary operators, correlation functions, radial quantization, the state–operator correspondence, and operator mixing. They then use the free scalar as a test case in which the dimension of ϕ^n is recovered by three complementary routes: Wick contractions, a flat-space saddle induced by the insertions, and a cylinder Hamiltonian/Bohr–Sommerfeld argument.

The main technical part develops a semiclassical description of heavy neutral operators in the double-scaling regime

$$n \rightarrow \infty, \quad \lambda \rightarrow 0, \quad \kappa = \lambda n \text{ fixed.}$$

The state created by a heavy operator is treated as a highly occupied state on the cylinder. Its leading energy is obtained from a periodic classical orbit, while the first quantum correction is expressed through the fluctuation determinant around that orbit. The derivation is organized through the resolvent, a path integral with periodic boundary conditions, Floquet theory for Hill operators, the Gel’fand–Yaglom theorem, and the Gutzwiller trace formula.

The final application is the critical $O(N)$ ϕ^4 theory in $d = 4 - \epsilon$. The homogeneous saddle is written in terms of a Jacobi elliptic function, the Bohr–Sommerfeld condition fixes the elliptic modulus as a function of λn , and the leading scaling dimension is encoded in a coefficient $C_0(\kappa)$. The quadratic fluctuations split into transverse and longitudinal Lamé problems, whose stability angles determine the one-loop coefficient $C_1(\kappa)$. The manuscript gives the resulting expansion

$$\Delta_n = nC_0(\kappa) + C_1(\kappa) + O(1/n),$$

discusses perturbative small- κ limits, matches known ϵ -expansion data, describes towers of derivative operators via occupation numbers, and identifies the onset of instabilities of the homogeneous saddle at larger λn .

General Evaluation

The manuscript is valuable and timely. It addresses a real pedagogical gap: the recent semiclassical treatment of heavy neutral composite operators uses a combination of CFT kinematics, classical mechanics, spectral theory, and renormalization that is not easily accessible from the original research papers alone. The text does a good job of building the reader’s toolkit before entering the interacting calculation. In particular, the progression from the state–operator correspondence to a cylinder energy problem, and then from the free oscillator to an anharmonic periodic orbit, is conceptually clear.

The technical presentation is also mostly convincing. The three free-theory derivations are a useful calibration of normalizations and of the semiclassical logic. In the interacting application, the chain of arguments is coherent: the Bohr–Sommerfeld condition fixes the modulus m , the classical

energy yields C_0 , the Lamé fluctuation operators yield stability angles, and the UV poles in the one-loop sums cancel against the renormalization of the classical action. The checks involving the free limit, the zero modes, the descendant mode with unit dimension shift, and the comparison with known perturbative data substantially increase confidence in the result.

The manuscript’s strongest contribution is as a pedagogical synthesis and as a road map through a technically difficult subject. It also appears to collect, extend, and systematize the operator-dictionary side of the calculation, including spin towers and occupation-number labels. For a high-energy theory journal that accepts substantial lecture-note or review-style articles, the paper is suitable in scope and likely to be useful to the community. I do not think a major reconstruction is required, but several points should be clarified before publication.

Major Comments

1. Clarify the precise novelty and intended genre of the article.

The manuscript is explicitly written as lecture notes, and much of the technical result is attributed to recent work on semiclassical composite operators. This is perfectly appropriate, but the introduction should state more sharply what is new in the present text and what is being reviewed. At present, the reader can infer that the novelty lies in the coherent pedagogical route, the unified derivation, the extension of some spectrum tables and spin-tower formulas, and the broader comparison with bootstrap, fuzzy-sphere, lattice radial quantization, conformal-window physics, and related semiclassical programs. It would be helpful to make this explicit in a short paragraph near the beginning.

This point matters for publication because the standards for a review or lecture-note article are different from those for a short research paper. If the paper is meant as a pedagogical review with some new extensions, the author should say so directly. If the paper contains new formulas not already present in the cited research articles, those formulas should be identified clearly, perhaps in the “How to Navigate These Lectures” section and again before the final $O(N)$ spectrum table.

2. State the domain of validity of the semiclassical expansion more sharply.

The double-scaling limit is introduced clearly, but some later claims about the three-dimensional Ising model and about the range where NLO semiclassics becomes the “state of the art” would benefit from more careful qualification. The controlled limit is large n , small ϵ , fixed en (or, equivalently, fixed $\lambda_* n$ in the Wilson–Fisher theory). The extrapolation to $\epsilon = 1$ and to moderate n is physically interesting and well motivated by the comparison plot, but it is not controlled in the same sense as the formal double-scaling expansion.

I recommend adding a concise discussion that separates three regimes: the strict double-scaling regime, the perturbative overlap regime in which the small- κ expansion can be compared with multiloop ϵ -expansion data, and the phenomenological $d = 3$ extrapolation. The statements around the missing $n = 6$ NLO point, the transition region $7 \leq n \leq 12$, and the claim that NLO semiclassics is expected to dominate for $n \gtrsim O(10)$ should be phrased as expectations supported by current comparisons, not as settled accuracy estimates. Including uncertainties or a table of the numbers used in the comparison would make this part more persuasive.

3. Give a little more detail on the renormalized one-loop determinant.

The derivation of C_1 is the technical heart of the manuscript. The presentation already contains the essential ingredients: transverse and longitudinal Lamé operators, stability angles, the zero

modes, the subtraction of the free contribution, the large- ℓ regularization, and the cancellation of the $1/\epsilon$ poles against the renormalized classical action. However, the final steps are compressed relative to their importance.

I suggest expanding the explanation of the regularized sums over stability angles. In particular, the functions $\sigma_1(\ell)$ and $\sigma_2(\ell)$ should be defined with enough precision that a reader could reproduce the subtraction prescription without consulting the original papers. The manuscript states that the first five divergent terms are subtracted in $d = 4$ and then added back by zeta regularization; it would be useful to display the asymptotic form being subtracted, at least schematically, and to specify how the zero modes and the $\ell = 1$ descendant channel are excluded or retained in each sum.

Similarly, the bare-coupling relation written in exponential form is efficient, but somewhat nonstandard in appearance. Since the pole cancellation is one of the most important checks of the calculation, the text should briefly connect this formula to the usual minimal-subtraction expansion $\lambda_0 = M^\epsilon(\lambda + \beta_0\lambda^2/\epsilon + \dots)$ and explain which terms are needed at the order considered. This would prevent possible confusion about whether higher-loop counterterms are being implicitly resummed or merely parametrized.

4. Clarify the treatment of zero modes and collective coordinates.

The manuscript correctly identifies the time-translation zero mode in the longitudinal sector and the $O(N)/O(N - 1)$ zero modes in the transverse sector. It also explains that zero modes are removed from the primed determinant and handled separately as collective coordinates. Since the final C_1 formula depends sensitively on these choices, the discussion would be stronger if the normalization of the collective-coordinate contribution were made more explicit.

For example, the reader should be told whether the collective-coordinate factors affect only an overall normalization of correlation functions, whether they contribute to the energy at the order being retained, and how this relates to the exclusion of the $\ell = 0$ modes from the stability-angle sums. This is especially important because the transverse zero modes are tied to a continuous internal symmetry direction, while the time-translation zero mode is tied to the periodic orbit itself. These two types of zero mode have different geometric origins, and the current text would benefit from spelling out why their treatment leads to the displayed C_1 formula without additional finite terms.

5. Make the operator dictionary and notation less ambiguous.

The occupation-number dictionary is one of the most interesting parts of the paper. It explains how excitations of Floquet modes map to derivative operators and how NLO semiclassics lifts the leading degeneracy. There are, however, a few points where notation could confuse readers.

First, the symbol κ is used both for the double-scaling coupling $\kappa = \lambda n$ and for the integer label of the Lamé operator ($\kappa = 1, 2$). Since both uses occur close together in the final calculation, renaming the Lamé index or warning the reader explicitly would improve readability.

Second, the operator degree n and the spherical-harmonic degeneracy n_ℓ are visually similar. The notation is standard enough, but a short reminder in the C_1 subsection would help.

Third, the relation between the Ising case, where ϕ^n can be discussed for general n subject to the \mathbb{Z}_2 selection rules, and the $O(N)$ singlet operator $(\phi_a\phi_a)^{n/2}$, which assumes even degree if n denotes the number of elementary fields, should be stated carefully. At several points the phrase “operator of degree n ” is used alongside $(\phi_a\phi_a)^{n/2}$. This is understandable, but for a reader tracking operator quantum numbers it would be better to define once and for all whether

n is the total number of elementary fields, the number of bilinears, or a semiclassical occupation number that is analytically continued between the two notations.

6. Streamline the broad outlook.

The final outlook is interesting, but it is much broader than the demonstrated results. The connections to conformal windows, asymptotically safe theories, entanglement, and wide neural networks are suggestive. Some of them are natural extensions of the semiclassical technology, while others are more speculative. For a journal article, I would either shorten this material or mark more clearly which directions are established applications, which are direct near-term extensions, and which are analogies. The neural-network paragraph in particular is likely to distract some readers unless it is made more concise and clearly framed as an analogy rather than a consequence of the analysis.

Minor Comments

- The phrase “an (heavy) operator” in the abstract should be changed to “a heavy operator”.
- In the conclusions, the phrase “cylinder energy of radius R ” should read “cylinder energy of radius R ”.
- The word “canovaccio” is evocative, but it may be unfamiliar to many readers. I recommend defining it the first time it appears or replacing it with “blueprint” in section and part titles.
- The discussion of Maslov corrections in the action-variable chapter is useful, but the final interacting calculation uses $I = 2\pi n$. The text should state explicitly why the Maslov shift does not alter the quoted large- n scaling dimension at the order retained, or how it has been absorbed into the definition of the relevant operator state.
- The comparison figure for the three-dimensional Ising CFT would be more useful if the numerical values were provided in an active table rather than only in commented material. This would make the benchmark easier to inspect and cite.
- The red entries in the $O(N)$ spectrum table should be checked for accessibility in black-and-white printing. A textual marker or a fourth column would be more robust than color alone.
- Some statements about scale invariance enhancing to conformal invariance should retain the usual caveats. The introductory chapter generally does this, but a few sentences read as more unconditional than necessary.
- Please check for small typographical and spacing issues throughout, such as doubled spaces, occasional missing spaces after punctuation, and repeated “noindent” commands near displayed equations.

Conclusion

I find the manuscript correct in its overall logic, useful in its exposition, and significant as a pedagogical synthesis of semiclassical methods for heavy neutral composite operators. The main derivation is supported by several nontrivial checks, and the final $O(N)$ application is sufficiently detailed to be valuable to readers who want to understand the recent literature without reconstructing

the calculation from scattered sources. The revisions requested above are mainly clarifications of scope, validity, and technical presentation. After these changes, I would support publication.