Referee Report

Weak-field waveforms for generic relativistic orbits

Summary

The manuscript proposes a Schwinger-Keldysh/worldline effective-field-theory framework for computing weak-field gravitational dynamics and gravitational waveforms for generic relativistic orbits. The central idea is to integrate out the gravitational field, but not the worldline degrees of freedom, and thereby obtain two objects: an in-in effective action for the worldlines, whose term linear in the advanced worldline variable gives the equations of motion, and a one-point function of the metric evaluated for arbitrary off-shell worldlines, which can later be evaluated on solutions of those equations. This separates the computation of the dynamics from the reconstruction of the waveform without expanding around straight-line scattering or quasi-circular motion.

The paper first reviews the classical limit of the Schwinger-Keldysh path integral in the Keldysh basis, emphasizing that the relevant classical diagrams contain one advanced field insertion and retarded propagation. It then specializes the formalism to point masses coupled to perturbative gravity in de Donder gauge, fixes the worldline parameter to proper time, and writes the metric path integral that defines the in-in worldline effective action. The computational section argues that, because the worldline couplings contain exponential factors, tensor integrals can be obtained by differentiating scalar Green-function convolutions rather than by performing standard integration-by-parts reductions. The manuscript lists the connected tree diagrams through order G^3, derives the leading inter-worldline interaction and a regulated self-interaction term, rewrites the order-G^2 scalar integral as a convolution of retarded Green functions, and gives the leading and next-to-leading ingredients for the asymptotic waveform at future null infinity.

The topic is timely and potentially important. A method that keeps the trajectories arbitrary while retaining relativistic retardation effects could provide useful input for gravitational-wave modelling outside the quasi-circular and small-velocity regimes. The paper is also conceptually attractive because it brings together the Schwinger-Keldysh formalism, post-Minkowskian worldline EFT, and position-space Green-function methods in a way that makes causality manifest. However, in its present form the manuscript is closer to a promising methodological note than to a complete technical paper. Several key derivations are compressed, some claims are stronger than the evidence displayed, and at least one leading-order formula appears to contain a notation or algebraic issue that obscures translation invariance.

Recommendation

I recommend publication after major revision. The proposed framework is interesting and plausibly publishable in a high-energy theory or gravitational-physics journal, but the manuscript should be strengthened before acceptance. The main revisions should clarify precisely what is new relative to existing Schwinger-Keldysh, MPM, multipolar EFT, and post-Minkowskian worldline methods; provide more derivational detail for the leading-order equations of motion and waveform; add at least one nontrivial benchmark against a known result; and correct or explain the apparent problem in the displayed leading inter-worldline force.

Evaluation of Correctness and Logical Structure

The logical structure of the paper is sound at the level of the broad formalism. The use of the Schwinger-Keldysh path integral is appropriate for causal classical observables, and the distinction between retarded and advanced variables is the right language for deriving classical equations of motion from an in-in effective action. The separation between an effective action for the worldlines and a metric one-point function is also natural. It resembles the division between source dynamics and radiation generation in the MPM and EFT approaches, but with the important difference that the trajectories are kept arbitrary rather than expanded around a special solution.

The manuscript is most convincing in its conceptual discussion in Sections 1 and 2 and in the qualitative explanation of the diagrammatics in Section 3. The causal flow of the diagrams in Fig. 1, the statement that only one advanced insertion is required in the classical limit, and the appearance of retarded Green functions are all consistent with the Schwinger-Keldysh framework. The hierarchy of scales used to justify point-particle gravity and the weak-field expansion is also reasonable.

The technical execution is less complete. Several derivations are presented as short sketches, and the reader is often asked to accept identities or simplifications without enough intermediate information to check signs, support conditions, normalization factors, or gauge-dependent terms. This is particularly important because the paper's central claim is not merely that the formalism exists, but that it simplifies actual waveform computations by avoiding integration-by-parts reductions and by reducing the problem to Green-function convolutions. To substantiate this claim, the leading examples should be transparent and internally checked.

Major Comments

1. **The novelty and scope of the method should be stated more sharply.**

   The introduction correctly situates the work relative to MPM, multipolar EFT, NRGR, post-Minkowskian scattering, self-force, and EOB approaches. However, the precise new contribution remains somewhat diffuse. The paper should state explicitly which ingredients are already standard and which combination is new. For example, Schwinger-Keldysh methods for radiation reaction and worldline EFT are well established; position-space retarded Green functions are also standard; and PMEFT already keeps classical worldlines visible in some computations. The distinctive point here seems to be the proposal to compute off-shell worldline equations of motion and asymptotic waveforms as two independent functionals of arbitrary relativistic trajectories, then solve the worldline problem only afterward. This should be made explicit early in the paper and revisited in the conclusion.

   It would also help to explain what the method is expected to deliver in a realistic application. Is the goal a numerical integro-differential evolution scheme for arbitrary weak-field binaries? A semi-analytic generator of waveform integrands to be evaluated on numerical trajectories? A route to EOB input? These are related but not identical deliverables. The present manuscript sometimes presents the framework as an alternative to EOB and sometimes as a possible ingredient for an EOB-inspired implementation. A clearer statement of the intended output would make the paper easier to assess.

2. **The derivation of the worldline effective action and the role of the advanced variables need more detail.**

Section 2 is conceptually clear but too terse for a paper whose subsequent formulae depend on the precise Schwinger-Keldysh bookkeeping. The expansion of the effective action in powers of the advanced worldline variables is written schematically, with terms linear in x_i^- producing the equations of motion and higher terms suppressed. The manuscript should explain why the omitted terms have the structure claimed, how the proper-time gauge choice is implemented on the two Schwinger-Keldysh branches, and how the normalization of the path integral affects disconnected diagrams.

The statement that the waveform may be computed independently from the equations of motion is plausible, but the derivation would benefit from a clearer distinction between three operations: integrating out the metric, expanding in advanced variables, and finally localizing the worldline path integral on the saddle. In particular, the metric one-point function is written with an insertion that can lie on either branch, but the branch independence in the classical limit should be shown or at least explained. Since this is one of the main conceptual claims of the paper, the reader should not have to infer the details from general Schwinger-Keldysh lore.

3. **The leading inter-worldline force should be checked and rewritten in a manifestly invariant form.**

   The displayed leading inter-worldline contribution in Section 3.1 is the first substantial test of the formalism. As written, part of the coefficient multiplying the derivative of the light-cone delta function contains terms involving inner products such as $\dot{x}_1 \cdot x_1$ and $\dot{x}_1 \cdot x_2$ rather than only the separation $x_1 - x_2$ and the two velocities/accelerations. This makes the expression appear non-invariant under spacetime translations. If this is a typographical issue, it should be corrected. If it is an intermediate expression in a particular coordinate convention, the convention should be stated and the final translation-invariant form should be given.

   This point is important because the leading inter-worldline interaction is the simplest place where the method can be verified. I strongly recommend rewriting this result in a compact form using $r^\mu = x_1^\mu - x_2^\mu$, $\dot{x}_1^\mu$, $\dot{x}_2^\mu$, and their derivatives, with all variables evaluated at the relevant retarded point. The manuscript should also show explicitly how the light-cone delta function is resolved, at least for a simple choice of parameter, and how the result reduces to the expected Newtonian acceleration in the slow-motion weak-field limit. This would provide a valuable consistency check.

4. **The self-interaction/radiation-reaction term requires a more careful discussion.**

   The treatment of the self-interaction diagram is one of the most delicate parts of the paper. The argument is that dimensional regularization moves the support of the retarded Green function inside the light cone, so the integral over the past worldline can be replaced by an expansion near the coincidence point, with finite nonlocal contributions suppressed by $d - 4$. This is a strong statement and should be justified more explicitly.

   The resulting local term contains third derivatives of the worldline and contractions involving $\dot{x}$, $\ddot{x}$, and $\dddot{x}$, with the normalization $\dot{x}^2 = 1 + \mathcal{O}(G)$ imposed for convenience. The manuscript should discuss whether this term is a genuine physical radiation-reaction force, a gauge-dependent off-shell term, a term removable by field/worldline redefinitions, or a combination of these. It should also clarify the treatment of mass renormalization and scaleless integrals. Since the gravitational self-force of a point particle is a subtle subject, the analogy with the Abraham-Lorentz-Dirac force in electrodynamics is helpful but not sufficient.

A useful revision would compare the local expression to known radiation-reaction or self-force structures in an appropriate limit, or else explicitly state why such a comparison is not meaningful before adding the inter-worldline contributions and imposing the equations of motion. Without this discussion, the reader cannot easily judge whether the self-interaction term is a physical result or merely an intermediate regularization artifact.

5. **The claim that integration-by-parts identities are bypassed should be qualified and supported.**

   The observation that tensor numerators can be generated by differentiating scalar Green-function convolutions is appealing and likely useful. However, the manuscript sometimes presents this as bypassing integration-by-parts identities in a broad sense. This should be stated more carefully. Differentiation with respect to source positions avoids tensor reduction for the tree-level examples shown, but it does not by itself solve all analytic difficulties at higher orders, especially when ultraviolet divergences, infrared logarithms, overlapping support regions, or boundary terms appear.

   The paper should specify the regime in which the replacement of momenta by derivatives is guaranteed to be sufficient. It should also make clear whether the scalar Green-function convolutions themselves are expected to be computed by direct position-space methods, differential equations, method of regions, or some hybrid strategy. The discussion near the end of Section 3.1 already mentions differential equations at higher orders; this should be integrated with the stronger claim about not needing integration-by-parts identities.

6. **The order-G^2 convolution formula needs more explanation of support and distributional details.**

   The scalar convolution of three retarded Green functions is a good example of the proposed method, but the derivation is too compressed. The formula involving Δ, the transverse variable ρ^μ, and the remaining delta function should be accompanied by a clearer discussion of the allowed kinematic regions. The paper states that the integration over ρ is trivial and gives a factor of π when either y_i^μ or y_j^μ is timelike. The boundary cases, sign conventions, and the role of the absolute value in the square root should be spelled out.

   This is not a request for a long appendix of routine algebra. Rather, the paper should provide enough detail that a reader can understand which parts of the convolution are ordinary functions and which are distributions, how derivatives with respect to worldline positions act on the theta functions and delta functions, and whether contact terms arise when sources approach the insertion point. These issues will become more important in the higher-order calculations that the paper advertises.

7. **The waveform section should connect more directly to gauge-invariant radiative observables.**

   Section 3.2 gives a simple and useful expression for the leading waveform at future null infinity and then the scalar integral entering the next order. However, the observable content of $\epsilon_{\mu\nu} h^{\mu\nu}(u)$ should be clarified. What gauge conditions are imposed at null infinity? Is $\epsilon_{\mu\nu}$ assumed to be transverse and traceless with respect to the asymptotic direction? How does this quantity map to the Bondi news, radiative multipoles, or the waveform conventions used in the scattering-amplitudes literature cited in the introduction?

   The next-to-leading asymptotic convolution is especially promising because it is much simpler than corresponding on-shell integrals. The manuscript should include at least one benchmark:

for example, specialize to straight-line scattering and recover the known leading waveform kernel, or specialize to a slow-motion bound source and recover the expected leading multipolar structure. A comparison of this kind would substantially increase confidence that the off-shell generic-trajectory expression has the correct normalization and support.

8. **The manuscript should distinguish preliminary results from completed results.**

   The introduction and conclusion state that state-of-the-art results will appear in a companion paper. That is acceptable, but the present paper should then be precise about what has been achieved here. The current results include the formal setup, the diagrammatic organization, the leading-order examples, and the scalar structure of next-to-leading integrals. They do not yet include full next-to-leading equations of motion, full next-to-leading waveforms, a numerical implementation, or a comparison with existing waveform models. The abstract and conclusion should avoid giving the impression that the paper already supplies a ready-to-use waveform model.

Minor Comments and Presentation Issues

1. The sentence "the diagrammatic expansion is a clear simplification compared with attempting a iterative solution of Einstein's equations" should read "an iterative solution."

2. In Section 3.2, the sentence "Similar computations, specialised to scattering trajectories, was performed" should use "were performed."

3. The notation alternates between x_i, x_i^+, and sometimes expressions without explicit plus superscripts after the Keldysh basis has been introduced. The paper should state when the plus superscript is being suppressed.

4. The proper-time parametrization deserves one more clarifying sentence. In an interacting spacetime, it is not immediately obvious whether τ_i is the proper time with respect to the full metric, the background metric, or an order-by-order gauge-fixed worldline parameter. This matters for identities such as $\dot{x}^2 = 1 + \mathcal{O}(G)$.

5. The integration limits in the leading inter-worldline term are written with an upper limit τ even though the source on the second worldline is parametrized by t. The causal meaning is clear, but the notation would be cleaner if the second-worldline retarded parameter and the first-worldline parameter were named distinctly.

6. The paper would benefit from a short table listing the order in G of each object: equations of motion, waveform, number of retarded source insertions, and which diagrams in Fig. 1 contribute. This would help readers parse the statement that the waveform at order G^L parallels the equations of motion at order G^{L+1}.

7. The acknowledgement that AI tools were used for coding, formatting, research, and writing is unusually broad. The authors should ensure that this disclosure is compatible with the target journal's policy and that all factual and bibliographic claims have been checked carefully.

8. Several references are very recent or forward-looking. The paper should make sure that all cited forthcoming or cutting-edge results are publicly available and that claims about "state-of-the-art" comparisons are phrased conservatively.

Conclusion

The manuscript contains a promising and potentially useful framework for generic weak-field relativistic waveforms. Its main strengths are the causal Schwinger-Keldysh organization, the decision to keep the worldline trajectories arbitrary, and the observation that position-space Green-function convolutions may provide a simpler route to off-shell dynamics and radiation kernels. These ideas are interesting enough to merit publication, but the paper should first make its leading examples more transparent, correct or clarify the apparent issue in the inter-worldline force, and add benchmark checks against known results. With these revisions, the article would be a valuable methodological contribution.