Referee Report

Higgsless Lagrangian SCFTs and Strongly Finite VOAs

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Summary and overall assessment

This manuscript studies four-dimensional $\mathcal{N} = 2$ conformal Lagrangian gauge theories with trivial Higgs branch and the vertex operator algebras associated with their Schur sectors. The motivating chain of ideas is attractive. The Higgs chiral ring embeds into the Schur sector, while the Higgs branch is conjecturally the associated variety of the Schur VOA. A C_2-cofinite VOA can therefore arise from a four-dimensional theory only if the latter is Higgsless. Conversely, the Higgs branch conjecture predicts C_2-cofiniteness for a Higgsless parent theory. The authors further argue that the vector-multiplet sector generically produces logarithms in the modular transform of the Schur index, leading to the conjecture that every Higgsless Lagrangian SCFT has a strongly finite but non-rational Schur VOA.

The first main part of the paper applies the Bhardwaj–Tachikawa classification to search for Higgsless theories. The initial filters are the absence of continuous flavor symmetry and the bound $n_h \leq n_v$. The authors then introduce a useful subquiver lemma: a nonzero F-term class supported on a subquiver remains nonzero in the full F-term quotient. They use explicit invariant contractions to eliminate loops, chain-trunk trees, high-valence trunks, and many trivalent cases. The resulting list in Section 3.1 and Table 2 consists of two sporadic theories whose Higgslessness is later supported by Higgs Hilbert-series computations, one infinite family of trivalent SO / USp quivers, and a third sporadic $\frac{1}{2}$**asym3**–USp(8)–SO(6) theory. For the last two entries the evidence consists of diagrammatic operator reductions, finite Hall–Littlewood indices in selected cases, and finite-order invariant searches.

The second main part proposes explicit Schur VOAs for the two established sporadic cases. For USp(4) with a half-hypermultiplet in the **16**, the proposed strong generators are the stress tensor, an odd $\mathfrak{sl}_2$ doublet of weight two, and an even triplet of weight three, at $c_{2d} = -28$. For SU(3) $\times$ SU(2) with a half-hypermultiplet in $\mathbf{8} \times \mathbf{2}$, the proposal has the stress tensor, two $\mathfrak{sl}_2$ triplets, and one singlet of weight three, at $c_{2d} = -30$. The authors bootstrap singular OPEs by imposing Jacobi identities, display low-weight null relations, infer nilpotency of the proposed strong generators in the C_2 algebra, and obtain closed forms and modular transformations for the vacuum characters. In each example a power of $\log q$ occurs in the S-transform of the vacuum character, which would establish non-rationality once the proposed algebra and character are identified with the physical Schur VOA and Schur index.

The subject is timely and the manuscript contains a substantial amount of valuable work. The systematic use of the Lagrangian classification is original, the two proposed logarithmic algebras are potentially important new examples, and the combination of OPE, C_2, index, MLDE, and modular data is impressive. These examples should be of real interest to both the SCFT/VOA community and researchers studying logarithmic tensor categories. The natural venue is JHEP.

However, the strongest claims currently go beyond what is established in the body of the paper. Two entries in the classification remain candidates; the universal nonvanishing arguments for the discarded families need stronger ideal-theoretic justification; and the two bootstrapped algebras have not been proved to equal the BRST Schur VOAs. In addition, the printed null-state data contain at least one relation that cannot hold in a nontrivial conformal VOA. These are correctness issues affecting the central conclusions rather than matters of presentation alone.

Recommendation

Major revision. The results are sufficiently novel and promising that I would encourage resubmission rather than rejection. Publication requires a precise separation of proved results from conjectural candidates, a correction and independent verification of the null-state/OPE data, and a more complete justification of the identification and strong finiteness of the proposed VOAs. If the authors choose not to supply the missing invariant theory and BRST arguments, a consistent weakening of the title, abstract, result statements, and conclusions would be an acceptable alternative for several of the points below.

Major comments

1. **The advertised classification is stronger than the result actually established.**

   The abstract states that the paper classifies all Higgsless Lagrangian SCFTs and then presents the infinite family and three sporadic cases as the resulting Higgsless theories. Section 3.1 is more careful: it calls Table 2 a complete list of *candidates*. Entries (i) and (ii) are supported in Sections 4.2.1 and 4.2.2 by the exact claim $\mathcal{I}_{\text{Higgs}} = 1$. By contrast, entry (iii) is explicitly based on a "strong but non-rigorous" argument and Hall–Littlewood calculations only for $m = 4, 8$. Entry (iv), the quiver in equation (3.12), has a partial contraction analysis, an enumeration only through conformal dimension eight, and the polynomial index $1 + 2t^5$. The distinction is scientifically important: these are two proved sporadic cases plus two conjectural classes, not yet four established Higgsless classes.

   The authors should either complete the invariant-theoretic proof for entries (iii) and (iv), or consistently state that they have classified a set of *candidate* Higgsless theories and have proved Higgslessness only for two members. This distinction should appear in the title, abstract, Introduction, the statement following Table 2, and Section 5. The scope of "all Lagrangian SCFTs" also needs a precise definition. The Bhardwaj–Tachikawa input classifies gauge algebras and matter representations, whereas the paper separately discusses discrete gaugings of free vectors. Please state whether disconnected gauge groups, discrete gaugings of interacting theories, global forms and line-operator choices, tensor products, and dual quiver presentations are included or excluded.

2. **The nonvanishing hypotheses used to eliminate broad quiver families are not yet proved.**

   The subquiver lemma in Section 3.3 is a useful and essentially algebraic statement, but it begins with an operator that is already known to be nonzero in the subquiver quotient. In several later applications, this hypothesis is replaced by the observation that the proposed contraction differs from an individual quadratic F-term. This does not establish non-membership in the *ideal* generated by all F-terms: an invariant may be a polynomial combination of those generators even when it is not itself one of them.

   This issue affects the loop operator in equation (3.8), the chain operator in equation (3.10), the high-valence operator in equation (3.23), and the trivalent operator in equation (3.25). The low-rank exception noted after equation (3.25), where trace identities make the proposed invariant vanish, demonstrates that such ideal consequences are not merely hypothetical. For each universal exclusion, the authors should provide an all-rank representation-theoretic proof, a Groebner or invariant-theory argument, or an explicit F-flat field configuration on which the proposed invariant is nonzero. All low-rank identities should be treated systematically. Without this step, completeness of even the candidate list remains plausible rather than demonstrated. Appendix D should also record enough rank, current, and attachment data for the reduction from the full branch classification to be auditable without reconstructing the entire argument from the earlier literature.

3. **The Hall–Littlewood criterion in equation (2.19) must not be stated or used as an unconditional equivalence.**

The manuscript itself acknowledges that polynomiality of the unflavored Hall–Littlewood index can result from cancellations between bosonic and fermionic contributions and that the argument invokes Higgs-branch geometrization. Equation (2.19) nevertheless displays an “if and only if,” and later polynomial indices are repeatedly described as evidence that an entire theory is Higgsless. A supertrace does not in general determine the dimension of the underlying graded ring. Moreover, the discussion immediately after equation (2.19) appears to interchange the two logical directions: the stated nilpotent-ring qualification concerns what can be inferred from a polynomial index, not the converse implication as labelled there.

Equation (2.19) should be replaced by precisely stated conditional implications or explicitly labelled as a diagnostic conjecture. The authors should say which hypotheses are proved for each application. In particular, the polynomial indices of the first two members of family (iii) and of entry (iv) cannot substitute for an all-rank Higgs-ring calculation. The exact meaning of “the index truncates” should also be stated: an exact evaluation as a polynomial is different from a finite series expansion through a chosen order.

4. **The two bootstrapped algebras have not yet been identified with the Schur VOAs of the gauge theories.**

   At the beginning of Section 4 the strong generators are inferred from low-order Schur and Macdonald indices, the resulting vacuum characters are matched to the gauge-theory Schur indices only through q^8, and the authors explicitly state that a direct BRST calculation would provide a definitive identification. Consequently, Tables 4 and 7 give well-motivated generator ansatze, but the manuscript does not exclude additional strong generators at higher weight or a different simple quotient with the same initial character. Likewise, matching the central charge and several coefficients does not by itself produce an isomorphism with the relative semi-infinite BRST cohomology.

   The preferred revision would construct the relevant cohomology classes and give a BRST or spectral-sequence argument that they strongly generate the full cohomology, including a proof that the proposed null ideal gives the physical simple quotient. For the $\mathrm{SU}(3) \times \mathrm{SU}(2)$ example, the presentation as an $\widehat{\mathfrak{su}}(3)_{-6}$ BRST reduction of $\widehat{\mathfrak{so}}(8)_{-2}$ seems particularly promising. If this is not feasible, all statements that the displayed OPE algebras *are* the associated VOAs, and hence that the two gauge theories have strongly finite non-rational VOAs, must be made conditional. The unconditional result would then be the construction of two candidate algebras with matching protected data to the displayed orders.

5. **Appendix F contains a null relation that is incompatible with a nonzero stress tensor.**

   Table 11 lists $T^{(5)}$ by itself among the weight-seven VOA null states required for the Jacobi identities of the $c = -30$ algebra. This cannot be a null state in a nontrivial conformal VOA. In state language $\partial^5 T$ is proportional to $L_{-7}|0\rangle$; applying L_5 and using $[L_5, L_{-7}] = 12L_{-2}$ would force $L_{-2}|0\rangle = 0$, contradicting both the existence of T and the nonzero central term in equation (4.40). Derivatives vanish after passing to the C_2 quotient, but Table 11 is explicitly presented as a list of VOA nulls used in Jacobi identities, not as relations only in $\mathcal{R}_{\mathcal{V}}$.

   The authors should identify whether this is a typographical omission from a longer relation, a confusion between a descendant and its C_2 class, or a problem in the Jacobi computation. Because the printed nulls underpin both associativity and the relations in Tables 5 and 8, the complete OPE and null data should be regenerated and independently checked after this correction. There is a second consistency point worth checking in Table 8: combining $2U_iV^i - W^2 = 0$ with both displayed relations $T^4 + 8TU_iV^i = 0$ and $T^4 + 8TW^2 = 0$ implies the stronger relations $TW^2 = T^4 = 0$, whereas the text only derives $T^8 = 0$. Either the stronger conclusion should be recorded or the coefficients corrected.

6. **The claim of strong finiteness requires more than the displayed C_2 nilpotency calculations, and the superalgebra conventions must be made explicit.**

Tables 5 and 8 give convincing finite sets of polynomial relations from which the *proposed* generators are nilpotent. This proves finite-dimensionality of the corresponding C_2 quotient only after completeness of the strong generating set and validity of the null ideal have been established. To call the algebra strongly finite, the manuscript also needs a clearly defined nonzero simple quotient of CFT type that is self-contragredient. Those properties may be automatic for a verified Schur VOA of a simple SCFT, but they are not automatic for an abstract algebra obtained from a finite OPE ansatz. The universal algebra, its quotient, and the logical dependence of each finiteness assertion should therefore be stated precisely.

There is also an important parity issue. The USp(4) proposal contains the fermionic generators Λ_a, and the symplectic-fermion examples in Section 4.1 are vertex operator *super*algebras. The paper should specify whether strong finiteness and rationality refer to the superalgebra, its even part, or an associated ordinary VOA, and should explain the relevant module category. The Schur index is a supertrace, so modular covariance may involve ordinary, twisted, and parity-twisted sectors. The applicability of the quoted C_2-cofiniteness and Miyamoto modularity statements should be given in the precise superalgebra convention used here.

7. **The modular and MLDE conclusions need analytic identities and a more careful distinction between solutions and module characters.**

   For both interacting examples, the MLDE is obtained by fitting an ansatz to a finite q series and then checking integrality to higher order. The closed forms for the vacuum characters are stated, but no derivation from the gauge integrals or proof of the required q-series identities is supplied. Once a closed form is known to be the exact physical Schur index, its explicit S-transform containing $\log q$ is a strong and essentially decisive non-rationality test. A finite-order fit alone does not establish that exact identity. The authors should prove the closed forms, cite a proof, or include an exact contour/constant-term derivation.

   Tables 6 and 9 also call the nonlogarithmic MLDE solutions characters of simple modules, while Section 5 lists determination of the simple modules as future work. An MLDE solution with integral coefficients need not be the character of an actual module. Please distinguish solutions, candidate characters, verified characters, pseudocharacters, and spurious solutions, and state exactly what is shown to lie in the modular orbit of the vacuum. The recent modular-invariance result for quasi-lisse characters by Arakawa, van Ekeren, and Li, already present in the bibliography, should be incorporated where relevant.

8. **The central computer-assisted claims should be reproducible, and the novelty comparison should be strengthened.**

   The proofs of Higgslessness for the two sporadic theories rely on Macaulay2 Hilbert-series calculations and gauge-invariant projection. The OPEs and nulls rely on Mathematica/OPEdefs, and the character and MLDE statements rely on high-order series calculations. Please provide ancillary scripts or notebooks specifying the polynomial rings, representation constraints, F-term ideals, gauge projection, Hall–Littlewood constant terms, Jacobi conventions, null-state search, and the orders through which all identities were tested. This is especially necessary in view of the impossible printed relation in Table 11. The exact Hilbert-series outputs $\mathcal{I}_{\text{Higgs}} = 1$ are among the strongest results of the paper and should not remain black-box assertions.

   Finally, the claim that the $c = -28$ and $c = -30$ algebras are new deserves a more substantive comparison than "to the best of our knowledge." The authors should compare central charge, generator weights and parity, outer automorphisms, vacuum characters, C_2/Zhu data, and modular representations with triplet algebras, symplectic-fermion orbifolds, logarithmic W-algebras, and known extensions, cosets, and BRST reductions. Relevant works already listed but not discussed include Kang–Lawrie–Song on indices and point schemes of Higgsless theories, Wang–Yan–Yang on Higgs branches and VOAs from IIB constructions, and Arakawa–van Ekeren–Li on modularity of quasi-lisse characters. A clear comparison would make the genuine novelty and significance of the proposed examples much easier to assess.

Minor comments

1. The Hall–Littlewood contour measures in Sections 4.2.1 and 4.2.2 are malformed typographically: the factors involving x_1 and x_2 run together. Moreover, the vector-multiplet product is said to run only over roots. The adjoint character also has rank-many zero weights, producing factors $(1-t)^{\operatorname{rank} G}$; these factors appear necessary for the stated polynomial answers and should be included or absorbed by an explicitly stated convention.

2. In equation (4.1), the second fermion is written at the origin while the denominator is $(z-w)^2$. Use either $\lambda^\beta(w)$ or a denominator z^2.

3. The description of Zhu's C_2 algebra in Section 2.2 should distinguish the two induced operations. Its commutative multiplication comes from the normally ordered (equivalently, (-1)-) product, while the simple-pole product induces the Poisson bracket; the commutative product does not come from the singular OPE as currently stated.

4. The group-theoretic argument in Section 4.2.2 uses generic simultaneous diagonalization of commuting matrices. Commuting complex matrices need not be simultaneously diagonalizable on nongeneric or nilpotent loci, which matter to the scheme-theoretic chiral ring. The claimed Macaulay2 calculation may cure this gap, but the text should not present the diagonal argument alone as a ring-level proof.

5. Please state explicitly whether the half-hypermultiplet matter contents have been checked for all relevant global gauge anomalies, and specify the global gauge groups used in the Hilbert-series projections.

6. The manuscript contains several correctable typographical problems, including "third sporadic theorym" in Section 5, "with the the Haar measure" in Appendix A, the caption grammar in Table 10, and mojibake in "Ramanujan–Serre" and in the apostrophe in "q's." The label named `eq;usp8Fterm` should also use the same colon convention as the other labels.

7. Several long formulae and tables would benefit from a notation check. In particular, distinguish consistently among four-dimensional dimension E, two-dimensional conformal weight h, and the power of the Hall–Littlewood fugacity. This would make the operator enumerations in Section 3.5.3 and the generator descriptions in Tables 4 and 7 easier to compare.

Novelty and significance

Subject to the corrections above, I regard the work as potentially important. The scarcity of explicit interacting logarithmic Schur algebras makes the two examples intrinsically valuable, and the $\mathfrak{sl}_2$ automorphisms, small strong-generating sets, nilpotent C_2 relations, and closed modular orbits provide unusually rich data. The classification program is also a useful bridge between four-dimensional Lagrangian gauge theory and the structure theory of logarithmic VOAs. A revised paper that states the candidate status accurately, supplies reproducible algebraic checks, and establishes or appropriately qualifies the BRST identification would meet the novelty and interest standards for publication in a high-energy theory journal.