Referee Report

“A note on Zamolodchikov’s recursion relation for the torus conformal block and its light limit”

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Recommendation: Major revision. The proposed A_2 light-limit recursion is potentially useful and appears algebraically coherent, but the manuscript does not yet derive its central two-scaling-sector construction with enough precision to establish the main result at journal standard. The novelty and practical advantages also need to be demonstrated more concretely relative to the existing formulas.

Summary of the paper

The manuscript studies the light, large-central-charge limit of one-point torus conformal blocks. Section 2 first reviews the Virasoro/Liouville Zamolodchikov recursion. With $\alpha = b\widetilde{\alpha}$ and $\lambda = b\widetilde{\lambda}$, the authors argue that only the $n = 1$ terms of the generic recursion in equation (7) survive. This gives the light recursion (18), from which they quote the known hypergeometric solution (21) and hence the Liouville block (22).

The new part concerns A_2 Toda theory. Section 3.1 reviews the generic $\mathcal{W}_3$ torus recursion (34), whose internal data are parametrized by the two Weyl-invariant combinations u and v^2. In the light limit, the physical block is written as $\widetilde{H}(\widetilde{\alpha}_2^2, \widetilde{\Delta} \mid q)$, with $\widetilde{\Delta} = 2\widetilde{\alpha}_1 + \widetilde{\alpha}_2$. The power counting in equations (41) and (42) selects the $n = 1$ and $n = 2$ residue families. The authors then introduce a coupled recursion for an auxiliary function h and the physical function $\widetilde{H}$ in equations (43) and (44). They solve the h recursion in terms of a Gauss hypergeometric function in equation (48), leading to the main recursive representation (49). The manuscript reports agreement through high order in q with the authors’ earlier instanton-counting expression.

The intended result is not a new conformal block: the same light A_2 block already has a shadow-formalism representation and is contained in the authors’ earlier general W_n construction. The contribution is instead a new recursive representation. Such a representation could be worthwhile. It organizes the expected degenerate poles directly, and it may be more convenient than formulas containing cancellations among spurious poles or ${}_3F_2(1)$ functions. In my view this is a reasonable subject for a concise technical paper, provided the derivation, comparison, and domain of validity are made substantially more explicit.

Assessment of correctness and logical structure

The Liouville power counting is convincing at the formal-series level. The exact denominator displayed in equation (15) makes clear why only $n = 1$ survives, and the residue (17) is consistent with the first coefficients in equation (19). The resulting expression (21) is also the expected global Virasoro torus block. However, the manuscript merely states that (18) can be solved to give (21). Since this known case is used as the primary validation of the limiting procedure, the missing verification should be supplied.

For A_2, the main formula is internally plausible. In particular, the leading coefficient in equation (42) is consistent with the generic pole formula, direct low-m expansions of equation (36) reproduce the residues in equations (46) and (47), and expansion of equation (48) agrees with the recursion (43) through q^{10} at generic rational parameters. I do not find an immediate algebraic contradiction in equation (49). The central logical gap is instead that the function h is never defined as a limit of the original function $H(v^2, u \mid q)$. Appendix A reveals that two inequivalent scalings of v^2 are being used. Without stating those two limits, equations (43) and (44), especially the term $h(0, \widetilde{\Delta} \mid q)$, appear without derivation. This is not a cosmetic omission: it is the mechanism by which the $n = 2$ residue family contributes to the physical block.

The checks quoted in the text are encouraging but insufficiently transparent. No coefficient generated by the main recursion is displayed, and the paper does not say precisely whether the high-order comparison is symbolic for generic parameters or numerical at selected values. Similarly, the claimed absence of spurious poles and the claimed advantage over the shadow formula are not illustrated. Thus the present evidence supports plausibility, but not yet a fully inspectable derivation and verification.

Novelty and significance

The finite-b $\mathcal{W}_3$ recursion (34) is taken from Ref. [29]. The target light block was obtained in a different form in Ref. [24], and the authors' own Ref. [35] gives a general W_n light one-point torus block whose $n = 3$ specialization is the block considered here. Equations (18)–(22) in the Liouville section reproduce known results. The manuscript's genuine novelty is therefore the limiting analysis of the finite-b recursion and the resulting representation (49), including the auxiliary hypergeometric seed (48).

That is a modest but potentially useful contribution for researchers working with conformal blocks, AGT relations, and Toda theory. Its significance rests mainly on computational organization rather than on a new physical object or a new spectrum. The paper can make a stronger case by showing explicitly how equation (49) exposes only the physical degenerate loci, how it compares term by term with the earlier representations, and in which parameter regimes it is practically superior. At present those points are asserted rather than demonstrated.

Major comments

1. **Define and derive the two v^2 scaling sectors.**

   From the definitions in Section 3.1 and the light scaling (38), one obtains the useful expansions

   $$u = \frac{3}{b^2} + 6 - 3\widetilde{\Delta} + O(b^2), \qquad v^2 = \frac{3^6 \widetilde{\alpha}_2^2}{b^2} + O(1).$$

   These equations should appear before the coupled recursion. They define the physical light function $\widetilde{H}(x, \widetilde{\Delta} \mid q)$ with $v^2 \sim 3^6 x / b^2$. However, an $n = 2$ residue of the generic recursion sends the shifted argument to $v^2 \sim b^{-4}$, as is visible in equation (54). The auxiliary function must therefore be a second limit, schematically

   $$h(\xi, \widetilde{\Delta} \mid q) = \lim_{b \to 0} H(v^2, u \mid q), \qquad \frac{b^4 v^2}{3^6} \longrightarrow \xi,$$

   with the corresponding expansion of u held fixed. In this sector the $n = 1$ terms vanish while the $n = 2$ terms remain finite. Conversely, when the physical b^{-2} sector first enters the b^{-4} sector, $b^4 v^2 / 3^6 \to 0$, which explains the seed $h(0, \widetilde{\Delta} \mid q)$ in equation (44).

The authors should give this derivation explicitly and trace the arguments after every residue. The first argument of h should be denoted by a neutral variable such as ξ, not by $\widetilde{\alpha}_2^2$, because in the b^{-4} sector it is a rescaled v^2 coefficient rather than the physical light charge invariant. Once this distinction is stated, the otherwise surprising structure of equations (43) and (44) becomes natural.

2. **Complete the power-counting argument leading from (34) to (43)–(44).**

   Equation (41) gives only the loose statement $R_{m,n} = O(b^{-4})$. The distinction needed for the derivation is sharper: equations (55) and (56) indicate $R_{m,1} \sim b^{-2}$ and $R_{m,2} \sim b^{-4}$, while equation (42) has exceptional denominator scalings for $n = 1$ and $n = 2$ and a b^{-6} leading behavior for $n \geq 3$. These facts should be assembled in one table or a short derivation, separately for the physical and auxiliary sectors. The paper should then show coefficientwise that all $n \geq 3$ terms vanish, rather than relying on the sentence that only $n = 1, 2$ contribute.

   This limiting operation is safest when stated as an identity of formal q-series: at any fixed power of q only finitely many pairs (m, n) occur. If an analytic interchange of the $b \to 0$ limit with the infinite recursion is intended, a uniform-convergence argument is needed. Appendix A should also define the shift c used in equations (53)–(56), specify whether it is fixed as $b \to 0$, and replace each ellipsis by a controlled $O(b^k)$ remainder. Finally, equation (45) should be restricted to $n = \pm 1, \pm 2$, the only cases used, or supplied with a branch convention for its fractional power.

3. **Strengthen the proof of the hypergeometric seed (48).**

   Appendix B compares the large-$\widetilde{\Delta}$ behavior and residues of the proposed solution with those of the recursion. This is the correct strategy, but the uniqueness step is unstated. The authors should formulate the proof coefficientwise in q. For generic ξ and $\widetilde{\lambda}$, each coefficient should be shown to be meromorphic in $\widetilde{\Delta}$, with a complete stated pole set; equations (59) and (60) should then establish equality of all residues, and the behavior at infinity should exclude an additional entire or polynomial term. The allowed range of the integer in equations (59) and (60) must be specified.

   The treatment of nongeneric parameters also deserves one sentence. Pole locations can collide, and Pochhammer factors can cancel residues. The authors should state that the formula is first established at generic values and then defined at exceptional values by meromorphic continuation, or give the appropriate limiting prescription. This would turn the present sketch into a complete proof of the seed used in the main result.

4. **Provide an inspectable verification of the main recursion.**

   At least the coefficients of q, q^2, and preferably q^3 generated by equation (49) should be displayed. They should be compared either with the light limit of the generic coefficients in equation (35), with a direct $\mathcal{W}_3$ descendant calculation, or with the earlier AGT expression after an explicit parameter dictionary. Such a comparison would test normalization, the hypergeometric seed, both residue families, and the shifted arguments in one place.

   The statement that agreement was checked "up to the twentieth order" also needs a precise description. The authors should state whether the comparison was performed symbolically for generic $\widetilde{\alpha}_i, \widetilde{\lambda}$ or numerically, which coefficients were retained, and which prefactors were removed. In the accompanying notebook the displayed difference is $O(q^{20})$, which normally means that coefficients strictly below q^{20} were compared. If the q^{20} coefficient itself was tested, the truncation and wording should be adjusted accordingly. A short table in the paper would make this central check reproducible without requiring the reader to reverse-engineer the notebook.

5. **Substantiate the pole and analytic-continuation claims.**

   Equation (49) has explicit pole loci

$$\widetilde{\alpha}_2^2 = \frac{4}{9}\left(\frac{\widetilde{\Delta}}{2} + m - 1\right)^2, \qquad m = 1, 2, \ldots,$$

   together with possible singularities of the hypergeometric seed when its denominator parameter is a nonpositive integer. The authors should identify these loci with the expected light-limit degenerate representations and explain how collisions or apparent seed singularities are handled. This is necessary to justify the statement that the new representation has no spurious poles.

   The comparison with Ref. [35] should exhibit at least one pole that appears termwise in the instanton expression, show its cancellation there, and show why it never appears in equation (49). Likewise, the discussion of Ref. [24] should state the convergence condition for the relevant ${}_3F_2(1)$ series and the parameter domain requiring analytic continuation. If the present result is being asserted only as a formal q-series, it does not by itself settle every analytic-continuation question; the manuscript should distinguish that formal advantage from convergence of the recursively generated series for $|q| < 1$.

6. **Clarify novelty and calibrate the literature comparisons.**

   The abstract and Introduction should state that the new result is a recursive representation of a known light block. A comparison subsection should give the normalization and parameter maps between equation (49), the finite-b recursion of Ref. [29], the shadow representation of Ref. [24], and the $n = 3$ specialization of Ref. [35]. It should identify precisely whether the new content is the $n = 1, 2$ survival mechanism, the two-sector recursion, the closed hypergeometric seed, or the resulting algorithm; in my reading, all four are aspects of a single new representation.

   The phrase "full agreement" should not be used for a finite-order series test. The claimed independent check of Ref. [29] should also be narrowed: it tests the partially degenerate light-limit sector and is compared primarily with the authors' own previous expression, not with the general finite-b recursion throughout parameter space. Statements that the new recursion is "particularly convenient" would be more convincing if supported by a coefficient-level example, operation counts, or timing for representative orders.

7. **Complete the Liouville validation and state the analytic status of both recursions.**

   Equation (21) is quoted without proof. A short substitution into equation (18), a contiguous-relation argument, or the same meromorphic residue method used in Appendix B would be enough. The known result should be cited directly at equation (21) or (22), not only discussed earlier in the Introduction. The normalization should also be reconciled with Refs. [22], [23], and [37].

   For both Liouville and Toda, the authors should state whether the recursions are identities of formal q-series, convergent series for $|q| < 1$ away from degenerate parameters, or meromorphic continuations thereof. The title "light Δ-recursion" in Section 3.3 is potentially confusing because the displayed denominators in equation (49) are poles in $\widetilde{\alpha}_2^2$ while $\widetilde{\Delta}$ is shifted recursively. The text should explain which internal quantum number is being analytically continued and why the terminology is appropriate.

Minor comments

1. Equation (2) defines $\mathcal{F}$, whereas equations (5), (20), and (22) use F without defining a second object. The notation should be made uniform.

2. Define $p_{12} = p_1 - p_2$ and $p_{23} = p_2 - p_3$ when these symbols first appear. The identity $u = 3(Q^2 - \Delta_{\boldsymbol{\alpha}})$ would also clarify the shift $u \mapsto u - 3mn$ in equation (34).

3. When equation (38) is introduced, restate $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \widetilde{\alpha}_3 = 0$ and indicate which two components are independent.

4. In equation (36), write $R_{m,n}(u)$, consistently with equation (34) and the surrounding text. The ranges and omission indicated by the primed product should be typeset so that there is no ambiguity about which product the prime modifies.

5. Define the Pochhammer symbol $(a)_m$ and state the genericity assumptions on all charge parameters before equations (17), (46), and (47).

6. State the normalization of the primary three-point matrix element implicit in equations (2) and (32). The recursions begin with $H = 1$, so the normalization convention should not be left implicit.

7. Equation (14) introduces a general object $\widetilde{R}_{m,n}$, although only $\widetilde{R}_{m,1}$ is subsequently used in the Liouville limit. Either give its general expression or avoid introducing the unused notation.

8. The residue notation in equations (59) and (60) should be replaced by the standard Res notation, with the variable of the residue shown explicitly.

9. The paper would benefit from a brief conclusion separating the known Liouville material, the new A_2 representation, the checks performed, and the open question of extending the method beyond A_2.

10. Bibliography entries that currently contain only a month and year should include their arXiv identifiers or complete publication information. In particular, Refs. [19], [28], and [35] are incomplete as printed.

11. Several prose edits are needed, including "consider a torus one-point block in which" in Section 3.1, removal of the redundant "Besides ... in addition" in the paragraph defining w, and correction of the comma splice immediately before equation (21).

Recommendation

I recommend **major revision**. The central recursion appears plausible and the two-sector mechanism is interesting enough to merit publication as a technical note. Before acceptance, however, the authors should define the auxiliary scaling sector, give a complete derivation of equations (43)–(49), make the meromorphic uniqueness argument precise, and provide explicit coefficient and pole comparisons with the existing representations. These changes would allow the reader to verify both correctness and novelty without having to infer the essential construction from the asymptotic formulas in the appendices.