

Referee Report

Cherednik integrable system: eigenfunctions at generic eigenvalues

arXiv:2607.09203

Recommendation: Major revision. The manuscript is not suitable for publication in its present form. The proposed factorized rank-reduction coefficients could be useful and may contain a genuinely new result, but the arbitrary- N theorem is asserted rather than proved, the construction of the advertised $N!$ branches is incomplete, several displayed identities are demonstrably false as written, and the novelty claims do not engage with directly relevant DAHA and Harish-Chandra literature.

Summary and overall assessment

The manuscript studies joint eigenfunctions of the commuting Cherednik operators in equation (23). Starting from the analogy with the ordinary Macdonald branching rule in equations (20)–(22), the authors expand an N -variable nonsymmetric Macdonald polynomial in the nonsymmetric Macdonald basis for $N - 1$ variables, as in equation (29). The corresponding one-variable coefficients are called skew nonsymmetric Macdonald polynomials. Explicit formulas are developed for the two ordering chambers at $N = 2$, followed by a substantial collection of $N = 3$ examples. Section 5 then proposes the factorized product formula (55) for partition-valued λ and μ at arbitrary N . Finally, the spectral labels are analytically continued, and the resulting infinite series are claimed to furnish generic Cherednik eigenfunctions with $N!$ branches and two polynomial reductions: nonsymmetric Macdonald polynomials at special eigenvalues and Baker-Akhiezer-type functions at $t = q^{-m}$.

There is a potentially interesting core idea. A proved, explicit $N \rightarrow N - 1$ branching formula for nonsymmetric Macdonald polynomials could give an efficient recursive construction and a useful bridge between polynomial eigenfunctions and chamber series. This would be relevant to specialists in DAHA, integrable difference systems, and related mathematical aspects of high-energy theory. The $N = 2$ and $N = 3$ calculations give useful evidence that factorization is present.

At present, however, the logical structure does not support the headline conclusions. The manuscript moves from selected low-rank examples to the general formula (55) with the phrase “straightforwardly generalizes,” but provides no proof. It then passes from the partition chamber to arbitrary compositions even though the only stated adjacent-swap rule is explicitly acknowledged not to survive repeated permutations. The analytic continuation is also not defined in a function space or formal completion, and the Baker-Akhiezer reduction is largely deferred. These are central issues of correctness, not matters of presentation alone.

Correctness and logical structure

The background definitions in Sections 2 and 3, together with the explicit $N = 2$ formulas in equations (34), (35), (40), and (41), make the intended normalization reasonably clear in simple cases. The difficulties begin when relations among deleted-variable expansions and different Weyl chambers are used as general identities. Several of these relations fail direct checks.

First, equation (26) does not reproduce the Hamiltonian defined in equation (17). Equation (17) is the elementary-symmetric Ruijsenaars Hamiltonian and should be related to

$$e_k(C_1, \dots, C_N)|_{\text{sym}},$$

whereas equation (26) contains $\sum_i C_i^k$, the power-sum Hamiltonian. These operators belong to the same commuting algebra, but they are not equal for $k > 1$; their eigenvalues are respectively $e_k(\Lambda_1, \dots, \Lambda_N)$ and $\sum_i \Lambda_i^k$.

Second, equation (33) has the wrong power of q . Applying equation (31) to the variables $(x_2, \dots, x_N, q^{-1}x_1)$ and then using the branching expansion gives

$$q^{\lambda_1+1}x_1^{-1}(q^{-1}x_1)^{|\lambda|+1-|\mu|} = q^{\lambda_1-|\lambda|+|\mu|}x_1^{|\lambda|-|\mu|}.$$

Thus the factor in equation (33) should be $q^{\lambda_1-|\lambda|+|\mu|}$, not $q^{|\lambda|-|\mu|+\lambda_1}$. The discrepancy is already visible for $N = 2$, $\lambda = (1, 0)$, and $\mu = (0)$: the coefficient on the left is one, while the printed right-hand side gives q^2 .

Third, the second line of equation (43) contradicts equation (41). The last numerator should be $(t; q)_{\mu-\lambda_1}$, as in equation (41), rather than $(qt; q)_{\mu-\lambda_1}$. For $\lambda = (0, 2)$ and $\mu = 1$, equation (41) gives $(1-t)/(1-qt)$, in agreement with the displayed polynomial $E_{[0,2]}$, while equation (43) as printed gives one.

There are further conceptual inconsistencies. Immediately after equations (36)–(37), the manuscript states that both $q = 1$ expressions are symmetric and that the Cherednik equations therefore trivialize. The second expression contains the factor $x_2^{\lambda_1+1}$ and is not symmetric in general. Moreover, the cyclic permutation in π remains even when the q -dilation becomes trivial. In Section 5, equation (57) is false without substantial support and strict-order restrictions. For example, take the interlacing partitions $\lambda = (2, 1, 0)$ and $\mu = (1, 1)$ with $i = 1$. Then $\sigma_1\mu = \mu$ and equation (55) is generically nonzero, whereas the prefactor in equation (57) contains $1 - q^{\lambda_2-\mu_2} = 0$.

The low-rank displays also need systematic rechecking. In Section 2.2, the displayed coefficient of $x_1x_2x_3$ in $E_{[0,3,0]}$ has a denominator containing $q^3t^2 + 1$. Combining the immediately preceding factorized terms instead gives the same numerator with $q^3t^2 - 1$. This is one concrete instance in which the decompositions advertised as equivalent are not equivalent as printed.

These errors may include correctable transcription mistakes, but their number and their placement in the central chain of reasoning prevent a positive correctness assessment. The revised manuscript should include a systematic symbolic or algebraic validation of every low-rank formula used to infer the general pattern.

Novelty and significance

The potentially novel part of the work is narrower than the abstract and conclusion presently suggest. The explicit factorization of the one-variable $N \rightarrow N - 1$ restriction coefficient, especially a proved version of equation (55) and its extensions to composition chambers, could be new and useful. The claim that this recursion is computationally efficient would also be valuable if supported by a precise algorithm and representative complexity comparison.

By contrast, generic nonsymmetric DAHA eigenfunctions with Weyl-group-indexed chamber expansions, together with their specializations to polynomial and Baker-Akhiezer regimes, have substantial prior literature. The revision must compare its functions and conventions directly with, at minimum:

- I. Cherednik, *On Harish-Chandra theory of global nonsymmetric functions*, arXiv:1407.5260. Theorem 2.2 constructs chamber series indexed by $w \in W$; in type A , $W = S_N$, giving $N!$ components. The rank-one formulas should be compared explicitly with the two series in that work.
- J. V. Stokman, *The c -function expansion of a basic hypergeometric function associated to root systems*, arXiv:1109.0613, which treats a nonsymmetric basic hypergeometric function as a meromorphic joint eigenfunction of the full nonsymmetric spectral problem.
- J. V. Stokman, *Connection coefficients for basic Harish-Chandra series*, arXiv:1208.6145, including the identification with multivariable Baker-Akhiezer functions and the termination at integral multiplicities.
- G. Letzter and J. V. Stokman, *Macdonald difference operators and Harish-Chandra series*, arXiv:math/0701218, and the related DAHA/qKZ work on asymptotically free solutions.

The nearby literature on nonsymmetric Pieri coefficients and generalized binomial coefficients, including W. Baratta, arXiv:0806.2695 and arXiv:1008.0892, should also be discussed. Those results concern fixed-rank multiplication and are not automatically the same as deletion of one variable, but the distinction and any transformation between the coefficients should be made explicit. Likewise, the phrase “introduce skew nonsymmetric Macdonald polynomials” needs qualification in view of existing uses of skew and interpolation constructions in the nonsymmetric setting.

Different gauges, normalizations, and spectral variables may mean that the present series are not literally identical to the cited Harish-Chandra series. That is precisely why a convention-by-convention comparison is essential. As written, the existence of $N!$ generic branches and the Baker-Akhiezer reduction cannot be presented as the main novelty. The authors should isolate and prove the genuinely new branching statement.

For a high-energy theory journal, the significance would also be clearer if the authors explained what the recursive coefficients add to the DIM/DAHA integrable-system picture beyond a new representation of known eigenfunctions. A nontrivial physical or computational application would strengthen the paper, but the primary requirement is a correct and new mathematical result.

Major revision requests

1. **State and prove the general branching theorem.** Equation (55) should be promoted to a precise theorem. The statement must specify the normalization of E_λ , genericity assumptions on q and t , the interlacing/support conditions on μ , and the exact class of partitions or compositions covered. A proof is required, for example by induction using DAHA intertwiners or by a direct calculation with all Cherednik operators. Agreement with $N = 2$ and $N = 3$ examples is not a proof at arbitrary N .
2. **Rebuild the composition-chamber construction.** Equation (57) needs corrected hypotheses and cannot serve as a general transition rule in its current form. For arbitrary N , the authors

must give formulas for general Weyl elements and prove independence of the chosen reduced word, equivalently the required braid consistency. If this cannot be done, the claims must be restricted to the partition chamber and to the explicitly established low-rank cases.

3. **Define the analytic continuation.** The paper defines $(a; q)_n$ only for nonnegative integer n , while equations (52), (55), and the subsequent continuation require negative or complex upper indices. The authors should define the chosen meromorphic continuation, state the domain of q , specify complex powers and logarithm branches, and list excluded polar hyperplanes. If the intended objects are formal series, the relevant completion and monomial cone must be stated instead.
4. **Prove the generic eigenfunction property.** The summation range in equation (60), the Weyl chamber in the x variables, and the convergence or formal meaning of the series must be given. The action of each C_i should be shown to be legitimate and to yield the eigenvalues claimed in equation (61). The spectral shifts for all other branches should be stated systematically. Analytic agreement on integer labels alone is insufficient without a rational-identity or coefficientwise argument.
5. **Correct and validate the formula chain.** Equations (26), (33), (43), and (57), the $q = 1$ discussion, and the inconsistent $N = 3$ displays must be corrected. The consequences of each correction for equations (44), (58)–(62), and the conclusion should be traced. A short ancillary file or appendix giving machine-checkable low-degree tests would be valuable.
6. **Establish or narrow the “triad” claim.** At $N = 2$, truncation at $t = q^{-m}$ is visible. At higher rank, the manuscript should prove finite support, identify the resulting function, and compare it with the established Harish-Chandra-to-Baker-Akhiezer specialization. If these properties remain future work, the second polynomial reduction should be presented as a conjectural direction rather than as a completed result.
7. **Reassess novelty against the direct prior art.** The two $N = 2$ branches in equation (62), the $N = 3$ series in equation (60), the asserted $N!$ family, and the Baker-Akhiezer specialization should each be matched, or shown not to match, the Cherednik and Stokman constructions cited above. The abstract, introduction, and conclusion should then be rewritten so that the genuinely new contribution is unambiguous.

Minor and presentation comments

1. Equation (29) writes $l_\mu \leq N$, whereas the text says that μ has $N - 1$ entries. State the indexing convention consistently.
2. Equation (31) contains qx_n where qx_N appears to be intended, and the composition in the right-hand side is missing a separator.
3. The eigenvalue equations alone do not define the normalization of E_λ at nongeneric parameters. State the monic triangular convention and the order being used.
4. Define precisely how $E_{\lambda/\mu}^{(k)}$ depends on the ambient rank and on the embedding obtained after

removing the k th variable. Equation (16), described as an associativity relation, should be formulated and proved rather than inferred from examples.

5. The claim that the recursion is the “fastest way” to calculate nonsymmetric Macdonald polynomials should be replaced by a neutral statement or supported by complexity estimates and benchmarks.
6. The appendix, printed as Section 8, needs substantial revision. Its relation to equation (55) should be explained, and the symbols Ξ , Ω , and a should be set up self-containedly. There are malformed expressions, including the denominator containing $\mathbf{x}\text{-I-}\mathbf{x}\text{.j}$, missing parentheses in several products, inconsistent η notation, and duplicated statements. These issues make the appendix difficult to use as evidence for the main result.
7. The prose and formulas require careful proofreading throughout. Examples include “Forth” for “Fourth,” “second large” for “second largest,” inconsistent uses of n and N , and numerous overlong displays. Breaking the central formulas into named propositions and short proofs would materially improve readability.

Recommendation

The factorized rank-reduction pattern is promising, and a rigorous theorem isolating that contribution could be of interest. The current manuscript, however, does not establish its general- N claims, contains explicit counterexamples to several printed identities, and substantially overstates novelty relative to existing nonsymmetric Harish-Chandra theory. I therefore recommend **major revision**. A revised version should be reconsidered only after the central branching formula is proved, the chamber construction and analytic setting are made precise, all formulas are revalidated, and the relationship to established generic DAHA eigenfunctions is explained in detail.