

Referee Report

Quantum field nucleating and Wigner functions

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Recommendation: Major revision. The Wigner-space formulation is promising, and the Affleck-like field-theory rate in equations (4.27) and (4.28) could be a useful result. I do not, however, think that the manuscript is ready for publication. The status of the stationary flux construction is not defined carefully enough, the central one-loop formula is presented without the derivation needed to verify its normalization and renormalization, the continuation through the breakdown found in Section 5 is not controlled, and the claimed disagreement with Linde is never exhibited formula by formula. In addition, the only illustrative potential contains a sign problem in its thermal correction and is not developed into the thermal application advertised in the abstract.

Summary and overall assessment

The paper begins from the probability for a field configuration to lie in a chosen metastable basin. Integrating the Moyal equation over conjugate momenta gives a continuity equation in configuration space and the flux-over-population identity (2.7). This identity is exact and, with the full Wigner function, can in principle describe both tunneling and over-the-barrier flow.

The calculation then specializes to a single real scalar. Section 3 expands the closed-time-path action about a static critical bubble. To quadratic order, the average field obeys the linear equation (3.8). The dividing surface is chosen to be $\phi_- = 0$, perpendicular to the unique unstable mode, and the exact flux formula is reduced to (3.13). Section 4 assumes a metastable state in approximate equilibrium. A one-loop Euclidean kernel about the static bubble is Wigner transformed to (4.12), and the factor

$$\sigma = \theta(\pi_- - \sqrt{|\lambda_-|} \phi_-)$$

selects the branch whose trajectories originate on the metastable side. The flux integral is then stated to give (4.27), or equivalently the periodic determinant expression (4.28).

There is a physically attractive core to this construction. For a quadratic unstable mode, the factor $\sqrt{|\lambda_-|}/[2\pi \sin(\sqrt{|\lambda_-|}/2T)]$ has the expected Affleck-like structure, and its high-temperature expansion is compatible with a classical transition-state limit. The manuscript also usefully draws attention to the fact that a static critical-bubble background does not imply that every fluctuation in its prefactor is classical.

The main difficulty is that the most consequential claims are not yet established at the level required for JHEP or PRD. Equation (2.7) is an exact instantaneous net-flux identity, while equation (4.27) is a one-loop, static-saddle, forward-flux approximation. The manuscript frequently moves between those two statements, and between a transition-state rate and a full decay rate, without making the necessary assumptions explicit. Moreover, the claimed differences from Affleck, Linde, and Langer cannot be checked because the comparison formulas are absent. These issues affect correctness, novelty, and the interpretation of the result rather than only its presentation.

Correctness and logical structure

The derivation of the integrated continuity equation in Section 2 and Appendix A is convincing: for a canonical scalar Hamiltonian, potential terms alter the momentum-space current but not the configuration-space current after integration over π . Thus (2.7) is a sound expression for the instantaneous signed flux through a specified boundary, divided by the instantaneous population of the metastable region.

The quadratic oscillator algebra behind (4.12) is also plausible. Above the first caustic of the Euclidean kernel, analytic continuation of a stable oscillator gives the cosine and tangent factors displayed in (4.23)–(4.26). At $\phi_- = 0$, integration over the selected half-line of π_- produces a finite factor proportional to $\omega_-/[2\pi \sin(\omega_-/2T)]$, with $\omega_- = \sqrt{|\lambda_-|}$. This is an important internal consistency check. However, the manuscript skips almost every field-theoretic step between that elementary integral and the determinant formula, including the zero-mode Jacobian, the treatment of the nonzero Matsubara tower of the unstable spatial mode, and the counterterm contribution.

The proposed validity window is more problematic. The Euclidean kernel used to derive the Wigner function develops a zero mode at (5.5), $T = \omega_-/\pi$, after which the Dirichlet fluctuation problem has a negative direction and the real Gaussian contour no longer defines the stated kernel. The final rate happens to remain finite until (5.4), $T = \omega_-/(2\pi)$, because a singular prefactor and a narrowing momentum integral cancel at one loop. That cancellation does not prove that the physical decay rate is the analytic continuation of (4.27) across the change of saddle structure. This is presently a conjecture, not a controlled result.

Novelty and significance

The potentially new contribution is not the abstract flux-over-population principle by itself, but its field-theory implementation around a static critical bubble and the resulting expression (4.27). The bibliography already cites Wigner’s transition-state work, yet the manuscript does not explain how (2.7), (3.13), and (4.18) relate to quantum transition-state and reactive-flux constructions. This comparison is needed to isolate what is specific and new in the field-theory result.

If the author can demonstrate that (4.27) is the correctly normalized real-time escape rate and that it differs from the conventional Linde prescription by a definite dynamical factor, the result would be interesting to the finite-temperature field-theory community. It could also be useful for cold-atom or other analog systems when T is not parametrically above the fluctuation scales. At present, however, neither significance claim is quantified: no determinant is evaluated, no ratio of rates is shown, and no experimental parameter point is analyzed. Since the difference is in a prefactor while the exponential usually dominates, a concrete benchmark is essential.

Major comments

1. **Define the quasi-stationary problem underlying the one-way flux.** Equation (2.7) is exact for the evolving density matrix of a closed system and gives a signed net flux. By contrast, Section 4 calls $W_{\text{nucl}} = \sigma W_{\text{eq}}$ time independent and uses it to obtain a nonzero one-way flux. A globally normalizable stationary Wigner state of the closed system has no loss of total metastable population. The construction therefore appears to be a quantum transition-state or quasi-stationary approximation with an absorbing boundary, an implicit source in the metastable basin, or both.

The revision should formulate this boundary-value problem explicitly. It should state the separation of relaxation and escape time scales, explain in what sense the slow overall normalization is omitted, and show how the local solution near the saddle is matched to the metastable distribution. It should also discuss recrossings and the dependence on the chosen dividing surface. Finally, the orientation of f_- must be fixed so that the positive argument of (4.18) unambiguously denotes the side originating in the metastable basin.

2. Repair the real-time initial-value derivation and delimit the channel that is calculated.

After writing $\phi_r = \phi_{cb} + \delta\phi_r$, equations (3.7) and (3.9) display integrations over both ϕ_r and $\delta\phi_r$, but no constraint enforces $\delta\phi_r(t_0) = \phi_r(t_0) - \phi_{cb}$. Equation (3.9) supplies the initial velocity through $\pi = \partial_t\phi_r$, but a second-order equation also requires the initial field value. If that condition is implicit in the path measure, it must be written explicitly; otherwise the displayed path integral double counts variables and does not define the claimed initial value problem.

This point is also where the paper should distinguish the exact statement that (2.7) can contain tunneling from the actual calculation, which uses a static bubble and linear over-the-barrier trajectories. The assertion following (4.7) that other Euclidean saddles simply provide additive rate components is not automatic: the rate is a ratio, saddle coefficients depend on integration contours, and static and time-dependent saddles can meet at a crossover. Either demonstrate a non-overlapping channel decomposition or restrict the main result to the static, over-the-barrier contribution.

3. Supply a complete derivation and renormalization of (4.27)–(4.28). The main result currently appears immediately after (4.23)–(4.26). A publishable derivation should show, at minimum:

- the ϕ_- and π_- flux integral and its contour;
- the normalization and orientation of the unstable eigenfunction;
- the conversion of each translational zero mode to a collective coordinate and the origin of $V(2T\Delta E_{cb}/\pi)^{d/2}$;
- the product identities that turn the spatial-mode factors into the periodic determinant in (4.28), including the nonzero Matsubara modes built on the unstable and zero spatial modes;
- all phases and factors of two associated with the negative mode; and
- a regulator and counterterm prescription demonstrating that the result is ultraviolet finite.

The statement near (3.2) that counterterms cancel the determinant divergence is not sufficient. In a semiclassical calculation the counterterm action evaluated on the bubble is part of the renormalized exponent/prefactor combination and must be specified. The author should also say consistently whether Γ is a total rate or a rate density; (4.27) is extensive in the spatial volume V , whereas most applications quote Γ/V .

4. Exhibit the Affleck, Linde, and Langer comparison in common conventions. This is the central physics claim of the paper, yet none of the three comparison rates is written. The revision should place the formulas side by side, preferably for Γ/V , and identify which determinants contain or omit the negative mode, how zero modes are normalized, what is obtained from the imaginary part of the free energy, and what real-time growth or transmission factor converts a statistical saddle weight into a decay rate.

An explicit $d = 0$ reduction of (4.27) to Affleck’s result would provide a valuable normalization check. The precise multiplicative ratio to the formula attributed to Linde should then be displayed.

Likewise, the high-temperature limit should be matched term by term to the Langer/EFT expression cited in Section 6. Because (4.28) itself has the form of a periodic Euclidean determinant, prose alone does not reveal where the claimed disagreement enters.

5. **Do not continue the result below (5.5) without a controlled argument.** At $T = \omega_-/\pi$, the $n = 1$ Dirichlet mode in Appendix B becomes a zero mode, the density-matrix saddle develops an additional negative direction, the difference-field fluctuations cease to be suppressed, and the quadratic Moyal treatment breaks down. Uniqueness of an asymptotic expansion in the region $T > \omega_-/\pi$ does not determine a physical answer across such a caustic or a possible Stokes transition. The one-dimensional Affleck result is useful evidence, but it is not a proof for a field theory with additional modes and competing saddles.

Unless the author supplies a uniform saddle analysis, an appropriate contour prescription, or a higher-order calculation, (4.27) should be claimed only in the region where its derivation is controlled. Section 5 should connect the temperatures (5.4) and (5.5) to the appearance of time-dependent periodic bounces and explain how double counting is avoided near the crossover. The broad claims in the abstract about the quantum-to-classical transition should be narrowed accordingly.

6. **Clarify the status of the high-temperature Langer rate.** Equation (4.18) selects the forward branch of the stable manifold for an isolated scalar with linearized Hamiltonian dynamics. A general Langer rate contains a dynamical growth or transmission coefficient and can be modified by damping, hard thermal modes, and recrossings. The manuscript itself notes after (6.7) that off-equilibrium particle effects are not captured by equilibrium field theory.

The author should state whether the present calculation fixes the growth coefficient to $\sqrt{|\lambda_-|}$ for the isolated relativistic scalar, and should derive the mapping to the EFT result of Gould and collaborators. If additional real-time physics is needed, (6.1) should be described as a transition-state contribution rather than as the universal full Langer rate. When the nonzero Matsubara modes are integrated into $\Delta F_{\text{eff,cb}}$ in (6.2), the revision should also specify which bubble profile, eigenvalues, and zero-mode normalization are derived from the effective theory, especially once the thermal correction is no longer perturbative.

7. **Replace the nominal example by a genuine, consistent benchmark.** The potential in (6.5), $V = m^2 \phi^2/2 - \lambda \phi^4/4!$, is unbounded for $\lambda > 0$ and does not contain the stable minimum assumed below (3.2). Moreover, with this sign convention the quartic coupling entering the tadpole is $-\lambda$, so the one-loop thermal mass shift in (6.6) should be $-\lambda T^2/24$, not $+\lambda T^2/24$. The magnitude of the perturbativity bound (6.7) may survive, but the stated effective potential and its physical interpretation do not.

More broadly, Section 6 does not perform the *simple thermal example* promised in the abstract. A revised paper should use a bounded metastable potential, calculate its critical bubble, negative eigenvalue, and renormalized determinant, and show the rate as a function of temperature. The same plot should compare (4.27) with the stated Linde formula and with the high-temperature EFT limit, and should mark the validity boundaries (5.5) and (6.7). If direct relevance to the cited analog experiments is retained, at least one set of experimental parameters should be mapped to the model. Otherwise the abstract should describe Section 6 as a parametric illustration rather than an application.

8. **Place the novelty and earlier Wigner work on a firmer footing.** The manuscript should distinguish the exact continuity identity (2.7), the quantum transition-state selector (4.18), and the

genuinely new field-theory determinant formula. It should compare these steps with the established quantum flux/reactive-flux literature, not only with Wigner’s original papers. Appendix C should also replace categorical judgments about earlier truncated-Wigner simulations with a controlled comparison of initial state, evolution equation, cutoff, critical configuration, and rate observable. The criticisms may be valid in a specified regime, but they do not presently establish quantitative disagreement.

Minor comments

1. Equation (3.10) has the wrong fluctuation operator as printed: $(-\nabla^2\phi_{\text{cb}} + V''(\phi_{\text{cb}}))f_-$ should be $[-\nabla^2 + V''(\phi_{\text{cb}})]f_-$, as in (4.14).
2. Equation (4.22) is dimensionally inconsistent. Its trigonometric arguments contain $\beta|\lambda_-|/2$ rather than $\beta\sqrt{|\lambda_-|}/2$, and the prefactor inside the error function also contains the wrong power of $|\lambda_-|$. This formula should be rederived from the half-line Gaussian transform.
3. Around (4.12)–(4.16), ϕ_r is used both for the full average field and for a fluctuation added to ϕ_{cb} . In particular, $W_{\text{cb}}[\phi_{\text{cb}} + \phi_r, \pi]$ and the expansion $\phi_r = \phi_{\text{cb}} + \sum_j \phi_{rj} f_j$ cannot both use the same convention. Introduce a distinct fluctuation symbol. Equation (4.19) must likewise act on the displacement from the bubble, not on the full field.
4. State explicitly the assumptions that the relevant static saddle has exactly one negative mode and only the d translational zero modes. The general claims about QFTs should not imply that the derivation already covers multifield, gauge, or constrained systems.
5. The discussion around (5.2)–(5.3) mixes exponential modes $e^{in\pi T\tau}$, odd anti-periodic modes, the Dirichlet sine basis used in (B.14), and the even periodic modes used in (4.28). Explain the boundary condition associated with each spectrum and how the odd modes cancel or reorganize in the final rate.
6. Below (6.1), a Matsubara frequency is of order $2\pi T$, while an eigenvalue of the quadratic fluctuation operator is of order $(2\pi T)^2$. The statement that $\lambda_{j,2n} \sim \pi T$ is dimensionally incorrect.
7. In (6.1), use d and $D = d + 1$ consistently and formulate the finite spectral correction as a regulated trace difference rather than an unexplained pairing of the N lowest spectra in two backgrounds.
8. The sign statement in the cosmology discussion in Section 7 should be checked. For fixed positive ΔE_{cb} , $d e^{-\Delta E_{\text{cb}}/T}/dT > 0$, not negative. If the intended statement relies on the temperature dependence of the critical bubble, that dependence must be included.
9. Several typographical errors impede the technical discussion, including *is compute explicitly* before (4.12), *possible possible* after (3.13), *Note the despite* in the following paragraph, and *based of* below (5.3). A careful proofread is needed.
10. The symbol V denotes both the scalar potential and the spatial volume. Use a distinct symbol such as $\mathcal{V}$ for the latter, especially in (4.27), (4.28), and (6.1).