Referee Report

$T\bar{T}$ deformation and multiple-flavor Lorentzian threads

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Recommendation: Reject in the present form. The proposed link between finite-cutoff generalized complexity and multiple-flavor Lorentzian threads is potentially interesting, but the two central results are not established. The Fefferman–Graham calculation does not support the claimed general hierarchy, and the vector-field construction neither proves the asserted local decomposition nor satisfies the Lorentzian-thread norm constraint. Repairing these points would require a new derivation and a reformulation of the main theorem rather than a conventional revision.

Summary of the paper

The manuscript studies generalized holographic complexity in a finite-cutoff bulk geometry intended to represent a $T\bar{T}$-type deformation of the boundary theory. Section 2 reviews a previous calculation for the complexity-volume prescription. Equations (6)–(9) compare a finite-cutoff volume with an undeformed expression and identify their difference with a Willmore functional proportional to $\int_\Sigma \sqrt{h}\, K^2$.

Section 3 attempts to extend that calculation to the *complexity = anything* functional. A general curvature density is expanded as $F_1 = 1 + \alpha(M_0 + \rho M_1 + \cdots)$, together with the near-boundary expansion (15) of the induced volume element. Equations (21)–(28) give the first two purported corrections. These are then extrapolated in equations (29) and (30) to a hierarchy

$$\delta\mathcal{C}_{\text{Any}} = \frac{1}{G_N}\sum_n \rho_c^{(3+2n-d)/2} W_\Sigma^{(n)}, \qquad W_\Sigma^{(n)} \propto \frac{1}{d-1-4n}\sum_I \int_\Sigma \sqrt{h}\, M_n^I K^2.$$

The paper identifies this hierarchy as its main result in Section 3.

Section 4 proposes a Lorentzian-thread interpretation. First, for a positive scalar density F_1, equation (35) conformally rescales the ambient metric so that the weighted hypersurface volume becomes an ordinary volume in equation (37). The authors then decompose $F_1 = \sum_I a_I F_I$, introduce weights $w_I = a_I F_I / F_1$, and seek flows

$$v_I = (w_I + \lambda_I)v + X_I, \qquad \sum_I v_I = v.$$

Proposition 1 observes that the fields are divergence-free if the compensators X_I are assumed to solve the required divergence equation. Proposition 2 is intended to establish such a construction locally by expanding the X_I in a frame of $v^\perp$. Finally, Section 4.4 interprets the different curvature sectors as distinct circuit layers or classes of quantum gates.

The intended synthesis is clear. The finite-cutoff calculation is meant to extend the CV and CA analysis of Ref. [10] to the generalized functionals of Ref. [9], while the second half seeks to connect that result to the Lorentzian-flow formulations of Refs. [11–14,18]. If made precise, an explicit weighted-flow dual for a nontrivial generalized complexity functional could be useful. Unfortunately, the present derivations do not support the stated conclusions.

Assessment of correctness and logical structure

There are two independent obstructions to publication. First, the radial integration in Section 3 is internally inconsistent. The signs and coefficients in equations (22) and (24) do not follow from the displayed integrands (21) and (23), and the all-n coefficient in equation (30) has the wrong dependence on n. The treatment also omits the logarithmic terms and exceptional dimensions that are already visible in equations (15)–(19). No explicit bulk solution is evaluated, so these formulae are not checked in even one nontrivial example.

Second, the flow construction does not prove the claim for which it is introduced. Proposition 1 assumes the central divergence equation rather than solving it. Proposition 2 has a dimension error, an ansatz that cannot represent general compensators, and mutually inconsistent signs. More decisively, equation (59) proves only that a putative v_I is timelike. It does not prove the bound $|v_I| \geq 1$ imposed in equation (42), and that bound is generically violated by the proposed decomposition. The final claim on page 15 that every sector is an admissible Lorentzian flow is therefore false.

There is also a structural mismatch between the two halves of the paper. The correction in Section 3 contains K^2, which depends on the candidate hypersurface. It cannot be absorbed into a fixed ambient conformal metric to which the standard min flow–max cut theorem applies. Moreover, a difference of two extremized functionals is later treated as the extremum of their pointwise difference. The manuscript acknowledges in a footnote that extremization is nonlinear, but the subsequent argument does not address the consequence.

Novelty and significance

The ingredients used separately are established in the cited literature: generalized geometric complexity functionals in Ref. [9], the finite-cutoff CV/CA calculation in Ref. [10], Lorentzian flow–cut duality in Refs. [11,12,18], and generalized or multiple-thread interpretations in Refs. [13,14]. The potentially new contribution is thus not the conformal identity for a positive weighted volume by itself, nor an arbitrary splitting of a vector field. It would have to be a valid dual program in which individual flow sectors are canonically associated with the curvature contributions generated by the deformation.

That association is not demonstrated. The decomposition of F_1 is basis-dependent, the weights need not be positive or regular, and the constant choice $w_I + \lambda_I = c_I$ used to obtain the claimed local construction removes the local F_I-dependence that was supposed to identify the flavor. The deformation coupling Γ appears in equation (5), but no quantitative relation between Γ, ρ_c, the weights, or the proposed flows is developed. Consequently the same formal splitting could be made for an undeformed weighted volume, and the paper does not establish that $T\bar{T}$-induced nonlocality produces the flavors. The circuit discussion and Figures 1–3 remain heuristic illustrations rather than evidence for a new physical result.

Major comments

1. **The difference of optimized complexities is not the optimization of the difference.**

   Equation (11) is said to include an implicit maximization, but equations (21)–(28) subtract the two expanded integrands as if the deformed and undeformed problems were evaluated on the same hypersurface. In general,

$$\max_{\Sigma} I_{\rm und}[\Sigma] - \max_{\Sigma} I_{\rm def}[\Sigma] \neq \max_{\Sigma}\big(I_{\rm und}[\Sigma] - I_{\rm def}[\Sigma]\big).$$

At first order in a clearly identified perturbative parameter, stationarity can allow the correction to the optimum to be evaluated on the unperturbed extremizer. That requires an explicit perturbation parameter, the variational equations, boundary conditions, and an accounting of when the displacement of the extremal surface first contributes. None of this is supplied, while equation (29) is presented as a general hierarchy.

There is a related issue before the subtraction is made. Equation (15) is the hypersurface expansion imported from the CV analysis, whereas the manuscript sets $F_1 = F_2 \neq 1$. The embedding then extremizes a weighted functional and generally obeys a different Euler–Lagrange equation; normal derivatives of F_1 and higher embedding coefficients can enter at the same cutoff order. The authors must derive the generalized embedding rather than assume that the maximal-volume expansion remains valid for an arbitrary complexity functional.

The same problem becomes explicit in Section 4.4. Equations (78)–(85) replace $\delta\mathcal{C}_{\rm Any}$ by a maximization of a functional involving $M_0 K^2$. Here K is the extrinsic curvature of the hypersurface being varied. Thus the proposed rescaled geometry itself depends on the unknown cut and cannot define the fixed Lorentzian manifold required by the flow–cut theorem. The authors must either derive a genuine convex dual for this higher-derivative surface functional or restrict the claim to a weighted volume whose weight is a fixed positive ambient scalar. The current use of the ordinary min flow–max cut theorem does not apply.

2. **The radial integration and equation (30) must be rederived.**

   The integrands in equations (21) and (23) contain positive ρG_{nn} terms, whereas the integrated expressions (22) and (24) contain them with a minus sign. Direct integration of the displayed $(\rho-\rho_c)K^2$ term also gives a coefficient differing by a factor of two from equations (27) and (28). The radial interval and orientation are not stated, but changing the orientation cannot repair the relative signs while retaining the printed leading term.

   Independently of this normalization problem, equation (30) cannot follow from equation (15). The relevant radial integral is

$$\int_{\rho_c}^{\infty} d\rho\, \rho^{\,n-(d+1)/2}(\rho-\rho_c) = \frac{4\,\rho_c^{(3+2n-d)/2}}{(d-2n-3)(d-2n-1)}$$

   when it converges. Therefore the coefficient inherited from the difference of the two K^2 terms has the denominator structure

$$\frac{d-3}{(d-1)(d-2n-3)(d-2n-1)}$$

   up to the overall normalization that must itself be fixed. It is not proportional to $1/[(d-1)(d-1-4n)]$. The two expressions happen to collapse to the printed patterns at the first low orders after cancellations, but disagree from $n=2$ onward. Since equation (29) is advertised as the main result of Section 3, an unsupported extrapolation from equations (27) and (28) is not sufficient.

   Equation (30) also omits the unity contribution $1/\alpha$ that appears in equation (27), and equations (30) and (31) omit the coefficients a_I used to define the curvature sectors later. These are not merely notational choices: they determine whether the $n=0$ formula reduces to the reviewed CV result and how each sector is weighted.

The revision would also have to treat the poles at special dimensions. Equations (15), (22), (24), (28), and (30) contain denominators $d-2$, $d-3$, $d-5$, and further dimension-dependent factors. At these dimensions the expansion generally develops logarithms rather than literal divergences. This is especially important because ordinary $T\bar{T}$ is a two-dimensional deformation, while the manuscript alternates between $T\bar{T}$ and generalized root-$T\bar{T}$ terminology. The $\rho^2 \log \rho$ term already displayed in equation (19) is simply dropped from equations (20)–(30). The range of d, the logarithmic branches, and the renormalization scheme must be stated.

3. **The curvature expansion lacks a defined bulk theory.**

   Equations (16)–(19) are presented as generic FG expansions, but the bulk field equations and matter content are never specified. In vacuum asymptotically AdS Einstein gravity,

$$R_{\mu\nu} = -\frac{d}{L^2} g_{\mu\nu}, \qquad R = -\frac{d(d+1)}{L^2},$$

   so R and $R_{\mu\nu}R^{\mu\nu}$ are fixed constants rather than the generic functions written in equations (16) and (18). If higher-curvature gravity or matter is intended, its equations must be given and the coefficients A, B, $\mathscr{C}$, and $\mathcal{A}$ derived. Before the convention $L = l = 1$ is introduced, equations (16) and (17) also omit the required powers of the AdS radius.

   The definitions of F_1 and M_n change within the section. Equations (12)–(14) use $F_1 = 1 + \alpha\, CI$, equation (20) expands CI, the text after equation (24) says instead $F_1 = M_0 + \rho M_1$, and equation (31) again omits both the constant and α. For the Weyl-squared example, equation (19) gives $M_0 = M_1 = 0$, so the first two displayed corrections do not test the example emphasized in the Introduction. The authors should choose a definite bulk model and a definite F_1, compute its coefficients through the claimed order, and verify the result against a direct calculation in an explicit geometry.

4. **The conformal flow representation is not established in the generality claimed.**

   The volume identity in equations (35) and (36) is valid only when F_1 is a smooth, strictly positive ambient scalar, defined independently of the trial hypersurface. The manuscript initially permits $F_1(g_{\mu\nu}; X^\mu)$, and later uses weights containing K^2. Lorentzian curvature contractions and arbitrary coefficients a_I need not be positive; $F_1^{2/d}$ and the weights (45) can consequently be singular or non-real. Precise positivity and regularity hypotheses are essential, not optional.

   Even under those hypotheses, the flow obtained in equation (38) is a field $\widetilde{v}$ obeying divergence and norm constraints in $\widetilde{g}$. Equations (40)–(77) abruptly use v, ∇, and $|\cdot|$ without specifying whether they refer to g or $\widetilde{g}$, or how the flow, divergence, norm, and boundary flux transform between the two geometries. The manuscript also assumes only that $\mathcal{M}$ is compact and oriented, whereas the Lorentzian flow–cut theorem requires causal and boundary hypotheses. These conditions and the normalization factors omitted in equation (38) must be restored before an equivalence can be claimed.

   There is an additional elementary error in Section 4.4. On a $(d-1)$-dimensional slice, the proposed rescaling $\widetilde{h}_{ab} = M_0^{1/(d-1)} K^{2/(d-1)} h_{ab}$ makes $\sqrt{\widetilde{h}} = (M_0 K^2)^{1/2}\sqrt{h}$, not $M_0 K^2 \sqrt{h}$ as required in equations (78)–(83). The metric factor would need the square of the printed conformal weight even before the more fundamental cut-dependence of K is addressed.

5. **Proposition 1 is conditional to the point of being tautological.**

   The desired conclusion is $\nabla_\mu v_I^\mu = 0$. Equation (54) assumes exactly the compensator equation that makes this conclusion true, so Proposition 1 does not establish that a compensator exists.

Equation (53) is circular because it calls λ_I a solution of an equation that already contains the unspecified X_I, and ∇X_I is not defined with an index. The proof in equation (60) also changes $w_I + \lambda_I$ into $a_I + \lambda_I$; the following text repeats this error in the summed condition.

To reproduce equation (44), equation (58) requires $f = 1$, equivalently $\sum_I \lambda_I = 0$ together with $\sum_I X_I = 0$. Allowing an arbitrary f and then saying that the result can be normalized changes the flux and the norm constraints and is not an innocuous rescaling of the original optimization problem. Boundary flux conditions are also absent. Proposition 1 should be recast as a simple algebraic lemma, while existence, boundary data, and admissibility must be proved separately.

6. **Proposition 2 is false as written and does not solve the constrained divergence problem.**

The ambient manifold in Section 4.1 has dimension $d + 1$. The orthogonal complement of a timelike vector therefore has rank d, not $d - 1$ as stated in equations (63)–(65). More seriously, equation (65) uses one common set of functions ψ^A for every flavor:

$$X_I = (w_I - \bar{w}) \sum_A \psi^A e_A.$$

This makes all X_I point in the same local direction and forces $X_I = 0$ wherever $w_I = \bar{w}$. It plainly cannot represent every orthogonal vector field, and it gives too few unknown functions to solve the independent source equations for general I. A general expansion would require flavor-dependent coefficients together with a separate constraint enforcing $\sum_I X_I = 0$.

There is also a direct sign contradiction: equation (67) has a negative right-hand side, equation (68) has a positive one, equation (70) returns to the negative sign, and equation (75) is positive again. The Poincare-lemma argument following equation (73) establishes at most a local solution of an unconstrained divergence equation. It does not simultaneously impose $g(v, X_I) = 0$, the common-coefficient ansatz, the sum constraint, future boundary data, and the norm bound. Finally, integrability of $v^\perp$ as a spacelike foliation is not by itself a global existence theorem for the stated PDE; global compatibility and boundary-flux conditions remain. Thus neither the local nor the asserted sufficient global construction has been proved.

7. **The individual thread norm and time-orientation constraints are violated.**

The paper defines every flavor to obey $|v_I| \geq 1$ in equation (42). With $X_I \perp v$, equation (61) instead gives

$$|v_I|^2 = (w_I + \lambda_I)^2 |v|^2 - g(X_I, X_I) \leq (w_I + \lambda_I)^2 |v|^2.$$

Condition (59) only makes this quantity positive; it does not make it at least one. For the authors' constant choice $w_I + \lambda_I = c_I$, take a saturated optimal flow with $|v| = 1$ and $X_I = 0$. If $\sum_I c_I = 1$ and there is more than one positive flavor, at least one $c_I < 1$, and hence $|v_I| = c_I < 1$. Adding a spacelike orthogonal X_I only decreases the Lorentzian norm.

Equation (62) contains the wrong sign for $g(v_I, v_I)$ and, even with the sign corrected, bounds ϵ only to preserve timelikeness. No condition ensures $w_I + \lambda_I > 0$, so future direction is also not guaranteed. The concluding claim that the construction yields $|v_I| \geq 1$ is therefore unsupported and in the simplest case contradicted. The authors need to state the exact multi-flavor convex program, including its joint norm constraint and objective, and prove that their fields are admissible for that program. A decomposition into timelike divergence-free vectors is not yet a Lorentzian-thread duality.

8. **The flavor assignment is neither canonical nor tied to the $T\bar{T}$ deformation.**

The same scalar F_1 can be decomposed into curvature terms in many bases, and total derivatives or algebraic curvature identities can change the list of sectors. The weights in equation (45) may be negative, may diverge where $F_1 = 0$, and are not invariant under such a basis change. Furthermore, setting $w_I + \lambda_I = c_I$ makes the longitudinal part of the proposed flavor independent of F_I; the remaining X_I is nonunique. The asserted map from a particular curvature invariant to a particular information channel is therefore not defined.

No worked background shows that increasing the deformation coupling changes the number, weights, fluxes, or nonlocality of the proposed threads. To establish the physical claim, the paper should study at least one explicit finite-cutoff solution and a positive density with two nontrivial invariants, construct admissible sector flows with boundary conditions, and compare their combined optimum with a direct evaluation of the generalized complexity. The relation to the construction in Ref. [14] should be stated at the level of the precise primal and dual programs. Without such a demonstration, Section 4.4 is a suggestive analogy rather than a result about $T\bar{T}$-induced nonlocal computation.

Minor comments

1. Equation (13) is labelled F_R but contains R^2, while equation (14) is labelled F_{R^2} but contains the Riemann tensor squared. Distinct notation such as F_{R^2} and F_{Riem^2} is needed.
2. The paper should distinguish ordinary two-dimensional $T\bar{T}$, its higher-dimensional analogues, and root-$T\bar{T}$. Equation (5) should be defined with the dimension and normalization for which it is intended.
3. Define the FG coordinate convention, the relation between ρ_c and r_c, and both limits of every radial integral written schematically as $\int_{\rho_c} d\rho$.
4. The roles and dimensions of H, h, G_{nn}, and K should be stated consistently. In particular, it is not always clear whether K belongs to the radial cutoff surface, the maximal bulk slice, or its intersection with the cutoff.
5. The factor V_x appears in equations (3) and (34), but not in equations (6), (11), or (37), even though the displayed integrals already include the spatial coordinates. The normalization and whether quantities are total complexities or densities should be fixed.
6. Section 4.1 alternates between a boundary time slice used for complexity and a subregion A with an entangling surface. State the correct anchoring and homology conditions for the Lorentzian complexity flow theorem rather than borrowing entanglement terminology.
7. Equation (38) suppresses the factors $V_x/(G_N L)$, the normal, and the measure in the flux. They should be retained in a purported equality.
8. In addition to the a_I/w_I error in equation (60), equation (76) uses u instead of v, equation (77) alternates i and I, and ∇X_I is repeatedly used where $\nabla_\mu X_I^\mu$ is meant.
9. The statement after equation (62) writes $g(v_I, v_I) = c_I^2 |g(v, v)| + \epsilon^2 g(Y_I, Y_I)$. The timelike term has the opposite sign in the convention used in equation (61).
10. The phrase complexity of formation before equation (29) is not the quantity defined in equation (25), which is a difference between deformed and undeformed prescriptions.

11. The cutoff expansion is described both as a finite sum to N in equation (29) and as an infinite sequence immediately afterward. The truncation rule and asymptotic meaning should be specified.

12. Equation (82) drops the factors ρ^n from the expansion stated in equation (31), and $v^0 > 0$ in equation (32) should be replaced by an invariant statement of future direction.

13. Reference [8] has an incomplete first author entry, and the bibliography should be standardized. The attribution near equation (40) points to Refs. [11,12] for a multiflavor formalism even though the manuscript elsewhere identifies Ref. [14] as the multiflavor construction. The manuscript also needs a careful language edit; several sentences in Sections 2 and 4 obscure the mathematical claim.

14. Figures 1–3 should define the meaning of each arrow and indicate which metric and slice are being used. At present they illustrate vector addition and a circuit analogy but do not validate local or global solutions.

Recommendation

The topic is appropriate for the holographic complexity community, and the idea of resolving a well-defined generalized functional into meaningful flow sectors could be worth pursuing. However, the current manuscript does not establish correctness of the finite-cutoff hierarchy, the flow–cut reformulation of the K^2 correction, or the admissibility and existence of the proposed flavor flows. These are the principal results, not peripheral details. I therefore recommend rejection in the present form. A future submission would need a fresh derivation, a correct multi-flow theorem with stated hypotheses and boundary data, and at least one explicit finite-cutoff example demonstrating a deformation-dependent physical effect.