Referee Report

"Circuit and Krylov complexity of primordial perturbations of modified gravity in inflation"

arXiv:2607.09408

Recommendation: Reject in the present form.

Summary and overall assessment

The manuscript studies two-mode squeezed curvature perturbations in canonical single-field inflation and in a proposed $f(\phi, R)$ modification. The authors first review Nielsen circuit complexity and the Lanczos/Krylov construction. They then attempt to derive quadratic Hamiltonians for the two inflationary models, solve for the squeezing amplitude r_k and angle ϕ_k, and use these solutions to compute circuit complexity, the quantity $K = \sinh^2 r_k$, Lanczos coefficients b_n, a diagonal coefficient c_n interpreted as dissipation, and Krylov entropy. The principal claims are that modified gravity changes squeezing near horizon crossing, raises circuit complexity, greatly enhances b_n and dissipation, and provides a more favorable setting for the growth of curvature perturbations and possibly primordial black-hole formation.

The topic could be of interest at the intersection of early-universe perturbation theory and quantum-complexity diagnostics. However, the present calculation does not establish its main results. The displayed quadratic actions are not derived from a well-defined background and gauge-invariant perturbation problem; the canonical action and potential expansion contain direct errors; the proposed $f(\phi, R)$ model is not actually specified; and the "slow-roll" reduction removes the potential contribution rather than implementing slow roll. The paper also conflates operator Krylov complexity with the spread of a state in the two-mode Fock basis, and it calls a decomposition of a Hermitian Hamiltonian an open-system construction without introducing a bath or a Lindblad generator. Finally, several headline conclusions contradict both $K = \sinh^2 r_k$ and the numerical curves. These problems affect the full chain from the action to every plotted complexity diagnostic. They would require a new derivation and new numerics, not a limited revision.

Correctness and logical structure

The intended logical sequence is clear: derive the perturbation Hamiltonian, obtain r_k and ϕ_k, and then evaluate complexity measures. The difficulty is that the first link in this sequence is not valid as written, while the later sections repeatedly change the mathematical object being evolved. In Section II A the evolution is that of an operator under the Liouvillian $\mathcal{L} = [H, \cdot]$. In Section IV B the purported Krylov basis is instead the set of state vectors $|n, n\rangle$, and the Hamiltonian itself is applied to them. Those are different constructions and generally give different tridiagonal problems. Likewise, the unitary closed-system Hamiltonian of Section III is divided into off-diagonal and diagonal pieces in Section IV B, with the latter reinterpreted as environmental dissipation. No dynamical principle justifies that reinterpretation.

The numerical analysis cannot repair these conceptual issues. Important parameters and initial data are absent, and several plotted statements are mutually inconsistent. In particular, Figure 1 gives a much larger late-time r_k in the canonical model, so equation (70) necessarily gives a much larger canonical K. Figure 6 indeed shows this, by roughly five orders of magnitude near the right edge, while the abstract, the end of Section IV, and the final conclusions variously claim that the $f(\phi, R)$ model amplifies K. Thus even the qualitative result to be assessed is not stated consistently.

Novelty and significance

The paper combines established two-mode squeezed-state circuit-complexity formulas with the cosmological Krylov framework already used in Refs. [13] and [14]. Reference [14], in particular, already studies inflationary closed- and open-system Krylov diagnostics, Lanczos coefficients, entropy, and modified early-universe dynamics. The identifiable new ingredient here is the replacement of the previous inflationary input by the inverse-symmetric scalar coupling motivated by Ref. [100]. At present, however, this coupling is inserted through the unproved quantity B rather than through a correct generalized-gravity perturbation action. The manuscript therefore does not yet demonstrate a new physical effect of $f(\phi, R)$ gravity as opposed to an effect of an assumed time-dependent mass term.

If rebuilt consistently, a comparison of circuit or spread complexity in a viable scalar-tensor inflation model could be useful. To establish significance for a PRD/JHEP audience, the authors would need to show that the result is robust under the complexity scheme, follows from a stable background compatible with the standard perturbation literature cited in Refs. [93–97], and has a controlled relation to conventional observables. None of these points is presently established.

Major comments

1. **The inflationary perturbation actions and Hamiltonians must be rederived.**

   Equation (26) is not the Einstein-scalar action in conformal coordinates for the metric in equation (21): the powers of a multiplying the kinetic terms are wrong and the time- and space-derivative signs are reversed for the stated signature. More decisively, the proposed quadratic action (28) contains $-\pi^2$ before π is defined, rather than the usual kinetic term in v'. If π is meant literally, equation (29) cannot follow by differentiating (28) with respect to v'. If it is a typographical substitute for v', the kinetic and gradient signs are still the opposite of the standard Mukhanov-Sasaki action. The Hamiltonian (30) retains negative kinetic and gradient terms and even integrates over $d\eta$, although a Hamiltonian at conformal time η should be an integral over the spatial slice only. Consequently equation (33), and hence the squeeze equations (38)–(43), do not follow from a controlled canonical system.

   The statement preceding equation (29) that the Einstein-Hilbert term cannot contribute because it has no explicit ϕ dependence is also incorrect. The curvature scalar contains the metric perturbations, whose constraint equations are essential in forming the gauge-invariant scalar mode. A simple consistency check exposes the sign problem: taking $A = 0$ and $z'/z = 0$ in equation (33) gives a negative oscillator energy proportional to $-k(N_k + N_{-k} + 1)$.

   There is also a direct algebraic error in the potential. For

   $$V(\phi) = \alpha\mu^2(1 - e^{-\sqrt{2/3}\,\phi})^2,$$

   the small-field expansion is $V = (2/3)\alpha\mu^2\phi^2 + O(\phi^3)$, not the constant $(2/3)\alpha\mu^2$ written in equation (27); the large-field limit is $\alpha\mu^2$. The definition $A = 4\alpha\mu^2 a^2/(3k)$ therefore has no demonstrated origin. The dimensions of ϕ, μ, and the exponential must also be specified.

The authors should start from a clearly defined gauge-invariant curvature perturbation, for example $v = z\mathcal{R}$, solve the lapse and shift constraints, display the correct normalization and effective mass, and perform the Legendre transform explicitly. The canonical limit should reproduce the standard action

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3x \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]$$

up to a stated canonical transformation. Until this check is passed, none of the subsequent Hamiltonian or squeezing results is reliable.

2. The $f(\phi, R)$ theory and its background are not specified.

Equation (44) defines a general function $f(\phi, R)$, but equation (45) gives only a dimensionless function $f(\phi)$. If the intended model is $f(\phi, R) = F(\phi)R$, that factor of R must be stated; if equation (45) is to be inserted literally into (44), there is no Einstein curvature term. The kinetic function $\omega(\phi)$ is never chosen. No background equations for H, ϕ, and R are solved, yet their time dependence is required in the perturbation action. Moreover, the flat-FLRW identity in cosmic time is $R = 6(2H^2 + \dot{H})$, not $R = 12H^2 + \dot{H}$ as stated below equation (44).

The transition from (45) and (46) to the quadratic action (47) is not shown. Differentiating the exponential coupling would retain exponential dependence on the background field, whereas the asserted $B = n^2a^2R/(\kappa^2\beta^2)$ contains none. The standard generalized-gravity analysis also changes the kinetic normalization and the Mukhanov-Sasaki quantity z, not only a mass term. Gauge constraints, the possible additional scalar degree of freedom, and ghost/gradient stability are not addressed. Finally, the approximation (46) discards one exponential and destroys the $\beta \leftrightarrow 1/\beta$ symmetry that motivated (45), without specifying the field range in which the truncation is valid.

A publishable comparison requires one explicit, dimensionally consistent model, its inflationary background, the complete second-order scalar action, stability conditions, and a demonstrated canonical limit. Equation (47) cannot be used as a substitute for that derivation.

3. The "slow-roll" reduction and numerical problem are not physically or mathematically defined.

Equations (43) and (53) are obtained by dropping the quantity A, which the paper identifies with the potential contribution. Slow-roll inflation is potential dominated; it does not mean that the potential term can be removed from the perturbation Hamiltonian. Background potential effects remain through H, ϵ, and z''/z. The statement after equation (43) that the dynamics becomes primarily kinetic and independent of V is therefore not a slow-roll result. The modified calculation compounds the issue by retaining B while eliminating A, without deriving a hierarchy that would justify this choice.

The numerical curves are not reproducible. No initial values for r_k and ϕ_k are given, although the equations contain $\coth(2r_k)$ and are singular at the natural unsqueezed initial condition $r_k = 0$. The paper gives only the inequality $n/\beta \geq 1$, not the values of n and β, and does not specify $R(y)$, $\phi(y)$, an integration interval tied to a background solution, a solver, tolerances, or convergence checks. The same symbol n is then reused as the Krylov-chain index in equations (63) and (69), making the modified b_n formula ambiguous.

There is a further background inconsistency: the exact de Sitter relation $a = -1/(H_0\eta)$ used to change variables has $\epsilon = 0$, while the canonical construction defines $z = \sqrt{2\epsilon}\, a$. A quasi-de Sitter background with nonzero ϵ, together with a well-defined Bunch-Davies initialization, is needed for this perturbation problem.

There is also a visible horizon-crossing inconsistency. With $H_0 = 1$ and $k = 0.01$, the stated de Sitter relation gives $k = aH$ at $y = \log_{10} a = -2$. The captions of Figures 1–3 say $y = -2$, while the vertical line in the plots is at $y = 0$ and the caption of Figure 6 instead says $y = 0$. The initial conditions and every numerical parameter must be listed, the horizon criterion must be used consistently, and data or code should be supplied so that all six figures can be regenerated.

4. The manuscript conflates operator Krylov complexity with state spread complexity.

Section II A introduces operator growth under $\mathcal{L} = [H, \cdot]$, but no operator inner product or initial observable is ever specified. Equation (3) normalizes every A_n by b_1 instead of by b_n, and equation (5) couples ϕ_{n+1} with b_{n-1} rather than the standard b_{n+1}. These errors already prevent the displayed review from defining a valid Lanczos chain.

In Section IV B, equation (55) simply identifies the Krylov amplitudes with the coefficients of the squeezed state in the Fock basis and takes $|n, n\rangle$ as the "operator" basis. These are state vectors, not operators in the Hilbert space on which the commutator Liouvillian acts. Applying H to $|n, n\rangle$, as in equations (61), (62), (67), and (68), is not the same as applying $[H, \cdot]$ to an operator basis. Equation (56), $K = \sinh^2 r_k$, is the mean pair number of the geometric Fock distribution and can be interpreted as a state-spread complexity in that chosen basis. It does not, without a separate construction, measure the Krylov growth of the curvature-perturbation operator.

The authors must decide which quantity the paper studies. For operator Krylov complexity they should name the initial curvature observable, define the inner product, construct the Liouvillian chain, and address the explicitly time-dependent Hamiltonian. For state spread complexity they should use that terminology and avoid claims about operator growth that have not been derived. This distinction is central, not semantic.

5. The claimed open-system and dissipation construction is not valid.

Equations (59)–(60) and (65)–(66) split a Hermitian two-mode Hamiltonian into pair-changing and number-diagonal pieces. A diagonal number term generates unitary phase rotation; it does not describe exchange with an environment. No bath, reduced density matrix, influence functional, master equation, Lindblad operator, or non-Hermitian effective generator is introduced. This is particularly problematic because Ref. [105], cited by the manuscript, formulates dissipative Krylov dynamics by modifying the Liouvillian with an actual Lindbladian and discusses why Arnoldi rather than ordinary Lanczos iteration is generally required.

The internal formulas are inconsistent as well. Equation (62) labels the diagonal action as H_{close} even though it comes from the term called H_{open} in equation (60). Equation (63) drops the $-k$ contribution present in (60) and (62), whereas equation (69) retains $-k$; hence (69) does not reduce to (63) when $B \to 0$. Equations (65) and (66) have unmatched or truncated brackets, and equation (67) changes the sign of the lowering term relative to the corresponding canonical action. Finally, the c_n displayed in (63) and (69) is imaginary, while Figure 5 shows a positive logarithm without stating that an absolute value has been taken. The assertion that $c_n \simeq b_n$ would imply negative growth of K is not derived from any evolution equation.

The open-system claims, Figure 5, and all conclusions about matter or energy exchange should be removed unless the authors supply a genuine open-system model and solve its appropriate Krylov/Arnoldi evolution.

6. The circuit-complexity construction is not self-consistent.

The Fock coefficient in equation (10) contains $\tanh^n r_k / \cosh^n r_k$ and lacks the squeezing phase, whereas the normalized two-mode state in equation (37) has the standard common factor $1/\cosh r_k$ and phase $e^{2in\phi_k}$. The coordinate-space expression in (10) is also singular as $r_k \to 0$, when it should approach the vacuum. Equations (12)–(13) call a Gaussian the reference state while giving its covariance matrix explicit dependence on the target parameters r_k and ϕ_k. Equation (14) transforms a wavefunction as $U\Psi U^\dagger$, which is a density-matrix transformation, and the text calls a general $GL(2,\mathbb{C})$ transformation unitary. The quantities Ω and ω in equation (18) are undefined.

Equation (20) also fails the vacuum check when read literally: at $r_k = 0$ its logarithm has denominator $1 - 2\cos(0) = -1$, rather than giving zero complexity. Its trigonometric arguments and parentheses must therefore be corrected before the plotted quantity can be identified with the displayed formula.

The authors should rederive equation (20) from a fixed reference covariance, a specified gate set and cost function, with all branches of the logarithm and arctangent stated. Because circuit complexity is scheme dependent, the absolute comparison between final values cannot be assigned physical significance without varying the reference frequency, gate penalties, and cost. The manuscript also calls the linear slope a Lyapunov exponent but provides no definition, fit interval, uncertainty, or relation to an independently diagnosed instability. Linear circuit-complexity growth alone is not evidence of chaos.

7. The headline numerical and physical conclusions contradict the formulas and figures.

The abstract says that the $f(\phi, R)$ coupling enhances the squeezing strength and that, because $K = \sinh^2 r_k$, this enhancement produces smaller growth of K. The stated implication is mathematically reversed: K is strictly increasing for $r_k > 0$. In Figure 1 the canonical curve reaches approximately $r_k \simeq 13.7$ at $y = 4$, whereas the modified curve is near 8. Figure 6 accordingly places the canonical K roughly five orders of magnitude above the modified result at late times. Section IV B initially acknowledges that canonical K is much larger, but its summary says all amplitudes are smaller in the canonical model, and the last bullet of Section V says that $f(\phi, R)$ significantly amplifies K. Section III and Section V likewise alternate between suppression and enhancement of entanglement and between less and more pronounced phase-space spiraling.

These are not small wording issues because they reverse the paper's principal conclusion. The authors must state separately what happens before and after horizon crossing, recompute every downstream quantity from one consistent solution, and align the abstract, discussion, and conclusion with the figures. The squeeze angle should be treated modulo its physical periodicity; comparing unwrapped values near 91 and 93 radians does not by itself quantify stronger phase-space rotation.

8. The claimed cosmological significance is not demonstrated.

The identity $K = \langle n_k \rangle$ for the selected squeezed-state distribution does not identify K with the curvature power spectrum, a local matter density, or a primordial black-hole abundance. No scalar spectrum $\mathcal{P}_\mathcal{R}$, spectral tilt, tensor-to-scalar ratio, transfer function, collapse threshold, or mass function is computed. Indeed, Section III says that the spectral index and tensor-to-scalar ratio are its focus, but neither appears in the results. The statements that increasing b_n tracks cosmic entropy, that $f(\phi, R)$ maximizes chaos, and that the plotted curves favor primordial black-hole formation therefore go well beyond the calculation.

The revised analysis should sharply distinguish a basis-dependent complexity diagnostic from observable perturbation statistics. If observational or primordial-black-hole claims are retained, the authors must compute the conventional gauge-invariant observables and show quantitatively how the complexity result adds information beyond them.

Minor comments

1. The scalar metric perturbation in equation (22) is not a general scalar-sector ansatz and its gauge is not stated. The lapse, shift, and spatial-curvature conventions should be defined before introducing the Mukhanov-Sasaki variable.

2. Equations (24) and (25), and the surrounding text, reverse the roles of the $F(R)$ and Einstein-Hilbert actions. The manuscript should use consistent notation for $F(R)$, $f(\phi, R)$, and $F(\phi)R$.

3. Equation (33) and equation (58) are assigned the same internal label, so cross-references can point to the wrong Hamiltonian. All labels and equation references should be checked after the derivation is corrected.

4. The identity quoted after equation (55) should read $\sum_{m=0}^{\infty} mz^m = z/(1-z)^2$ for $|z| < 1$; the factor m and the summand are missing in the text.

5. Figure 2 is a plot of ϕ_k, but its caption calls it a plot of r_k. The discussion refers to fluctuations near $y = 3$, although the visible transient is near $y = -3$. Figure 5 has "ffective" in its title.

6. For Figures 4 and 5 the authors must state the fixed Krylov index used in the time plot, distinguish it from the exponent parameter in equation (45), and say whether the ordinate is $|c_n|$, $\operatorname{Im} c_n$, or another quantity.

7. The numerical values quoted for the final circuit complexities in Section IV A do not match the endpoints visible in Figure 3. Values should be generated directly from archived numerical data and quoted with an appropriate precision.

8. The paper repeatedly says that modes "exit the event horizon." The relevant condition is Hubble-horizon crossing, $k = aH$, not exit from a cosmological event horizon.

9. The relation between the rotation angle θ_k in equations (34)–(35) and the phase entering the circuit reference/target states should be stated. Dropping an overall phase in the state does not automatically remove all phase dependence from a chosen circuit geometry.

10. The presentation contains substantial repetition, including nearly duplicated paragraphs in Section IV A and the summary. A thorough rewrite should shorten the general introductions and devote space to the missing derivations and reproducibility information.

Recommendation

The manuscript addresses an interesting general direction, but its main results are not presently correct or sufficiently defined for publication. The inflationary perturbation theory, Krylov construction, open-system interpretation, and numerical conclusions would all have to be reconstructed. Because these changes would alter the central calculation and likely the qualitative claims, I recommend rejection in the present form rather than major revision. A new submission based on a fully specified stable scalar-tensor model, a correct gauge-invariant quadratic action, a consistent choice between operator and state complexity, and reproducible numerics could be evaluated on its own merits.