Referee Report

"Factorization Algebras and Quantum Groups from Generalized Poisson Sigma Models"

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Recommendation: Major revision.

Summary of the manuscript

The manuscript develops a broad field-theoretic framework intended to connect shifted Poisson geometry, holomorphic–topological quantum field theory, factorization algebras, extended defects, and quantum groups. Its organizing object is a generalized Poisson sigma model on $\mathbb{R}^{d+1} \times \mathbb{C}^m$ whose target is described by what the author calls a shifted chiral Poisson algebra. In the topological specialization, the construction starts from a $(1-d)$-shifted Poisson algebra and produces a bulk theory with complementary Dirichlet and Neumann boundary conditions. The proposed dictionary identifies the Dirichlet boundary observables with the original Poisson algebra, the Neumann boundary observables with a Koszul-dual object, and bulk observables with a suitable higher Hochschild center. Interfaces, enriched boundaries, modules, coisotropic boundary conditions, and higher-codimension defects are then described in terms of morphisms and modules over the target Poisson data.

Sections 3 and 4 use perturbative boundary diagrams to motivate deformation quantization and quantum-group structures. For Lie and quasi-Lie bialgebra examples, the author relates boundary algebras to quantized universal enveloping algebras, associators, and R-matrices. The Chern–Simons discussion recovers the familiar coefficient $1/24$ in the leading associator term and formulates conjectures concerning universal line algebras. Section 5 extends the construction to m holomorphic directions by introducing the $cP_{d,m}$ formalism and multivariable higher $\boldsymbol{\lambda}$-brackets. Section 6 surveys a substantial range of examples, including holomorphic–topological twists of supersymmetric gauge theories, a Virasoro-type model, four-dimensional Chern–Simons theory, and a proposed Yangian construction from a boundary of a five-dimensional twist. Section 7 gives an abstract "universal bulk" construction and heat-kernel convolution formulae that are meant to explain the preceding boundary and corner calculations uniformly.

The conceptual ambition is considerable. The paper brings together several bodies of work that are often presented separately: AKSZ and shifted-cotangent constructions, the Butson–Yoo universal bulk, perturbative factorization algebras, chiral Poisson structures, Koszul duality, and the field-theoretic realization of quantum groups. The potentially most original parts are the multiholomorphic $cP_{d,m}$ formalism, the organization of extended objects within generalized Poisson sigma models, the universal bulk-to-boundary and bulk-to-corner identities, and the proposed realization of Yangian data on the boundary of a five-dimensional theory. These ideas are interesting and could be valuable to the high-energy theory and mathematical-physics communities.

At present, however, several central definitions and calculations are internally inconsistent. Some of these are normalization or indexing errors, but they occur precisely in the results that support the manuscript's main claims. In addition, a number of statements advertised as constructions or recoveries are established only formally, at leading perturbative order, or are explicitly conjectural later in the paper. The manuscript therefore does not yet meet the standard of correctness and claim calibration expected for publication in JHEP or PRD. I would nevertheless encourage a substantially revised version because the underlying synthesis appears promising.

Assessment of correctness and logical structure

The broad logical progression is natural: define the classical target data, construct a bulk theory and its boundary conditions, calculate boundary operations, pass to quantum examples, and finally explain the common propagator mechanism through a universal-bulk theorem. There are also many useful examples that help relate the abstract language to familiar theories. The difficulty is that the manuscript frequently moves from a suggestive diagrammatic calculation to an all-orders algebraic conclusion without isolating the hypotheses or intermediate propositions needed for that conclusion. This is especially consequential in Sections 3, 5, 6, and 7.

The central $cP_{d,m}$ definition is not well-defined with the indices and signs currently printed. The heat-kernel proof that supplies the normalization for the boundary operations contains a factor-of-two error, while different sections use mutually incompatible normalizations. The explicit Yangian coproduct coefficient in the main text is contradicted by the appendix calculation intended to prove it. A proposed extended-defect coupling also appears not to satisfy the classical master equation for a general Poisson target. These issues prevent the principal conclusions from being checked as written.

The manuscript also needs a sharper separation among five levels of result: rigorous theorem, formal perturbative identity, tree-level check, conjectural identification, and review or synthesis of known material. This would not diminish the paper. On the contrary, it would make the genuinely new results much easier to recognize and evaluate.

Major comments

1. **The definition of the shifted chiral Poisson structure in Section 5.1 must be made internally consistent.**

   Definition 5.1 is the foundation of the holomorphic–topological generalization, but its displayed arities do not agree. The definition of $\mathrm{cPol}(A)$ uses maps with k inputs, the variables $\boldsymbol{\lambda}^0, \ldots, \boldsymbol{\lambda}^{k-1}$, and S_k-invariants. Immediately afterward, an element is described as a map on $a_0, \ldots, a_k$ with variables $\boldsymbol{\lambda}^0, \ldots, \boldsymbol{\lambda}^k$. Equations (5.4)–(5.5) and Remark 5.1 likewise use $k+1$ inputs and an S_{k+1} action. Equation (5.4) even combines the inputs $a_0, \ldots, a_n$ with a list of variables ending at $\boldsymbol{\lambda}^k$. This is not merely a cosmetic shift of notation: the arity determines the grading, symmetry, Maurer–Cartan equation, and coefficients entering the action (5.11).

   There is a second internal conflict in the strict binary axioms. The displayed sesquilinearity rules state
   $$\{\partial_i a_{\boldsymbol{\lambda}} b\} = -(\partial_i + \lambda_i)\{a_{\boldsymbol{\lambda}} b\}, \qquad \{a_{\boldsymbol{\lambda}} \partial_i b\} = \lambda_i \{a_{\boldsymbol{\lambda}} b\}.$$
   Remark 5.1, however, defines the binary bracket by substituting $\boldsymbol{\lambda}^0 = -(\boldsymbol{\partial} + \boldsymbol{\lambda})$ and $\boldsymbol{\lambda}^1 = \boldsymbol{\lambda}$ into a multilinear operation satisfying $X(\ldots, \partial a_i, \ldots) = -\lambda^i X$. That prescription yields the opposite assignments for the two inputs. The ordinary Poisson vertex algebra convention also places $-\lambda_i$ on the first input and $(\partial_i + \lambda_i)$ on the second, up to an explicitly declared alternative convention.

In addition, the signs in graded skew-symmetry and the Leibniz rule should be rederived with the full $(1-d-m)$ shift rather than imported with a d-dependent exponent only.

The author should replace this material with one consistent operadic definition, state the cohomological degree of every k-ary operation, and derive the strict binary rules from it. Proposition 5.1 then needs a proof, not just the statement that the argument is the same as in a previously cited setting: at minimum, the proof should verify that the insertion bracket closes on the sesquilinearity quotient, has the asserted degree and signs, and that the Maurer–Cartan equation gives the classical master equation for (5.11). Until this is done, the higher $\boldsymbol{\lambda}$-brackets and all examples depending on them cannot be assessed.

2. **The bulk-to-boundary normalization in Theorem 7.1 and its uses must be recomputed globally.**

   Theorem 7.1, equation (7.24), states that the convolution of two reflected bulk propagators is $\frac{1}{2}P_{\mathcal{E}}$. The displayed calculation does not establish this consistently. Before equation (7.26), integration over the normal coordinate produces a numerator of the form

   $$u_1 Q_x^\dagger + u_2 Q_y^\dagger$$

   divided by $(u_1+u_2)2\pi\sqrt{u_1u_2}$. Lemma 7.1 identifies the two differentiated heat kernels, so the factor u_1+u_2 cancels and leaves one copy of $Q_x^\dagger K/(2\pi\sqrt{u_1u_2})$. Equation (7.26) instead writes $(Q_x^\dagger + Q_y^\dagger)K/(2\pi\sqrt{u_1u_2})$, which is twice that expression. Equation (7.27) silently returns to the one-$Q^\dagger$ expression. With that corrected intermediate step, the beta integral can indeed give the stated factor 1/2; with equation (7.26) as printed, it cannot.

   More importantly, the surrounding text does not use a single convention. Equation (1.2) states that the same convolution is $P_{\mathcal{E}}$, not $\frac{1}{2}P_{\mathcal{E}}$. Section 5.3 again invokes the result with unit coefficient immediately after equation (5.14). In the purely topological discussion, the coefficient in equation (3.3) gives twice the unit flux on the sphere, whereas the Stokes argument in Proposition 8.1 assumes unit flux. Section 4 introduces an additional half-propagator convention for the Chern–Simons example. Because these constants feed directly into the boundary bracket, coproduct, R-matrix, and associator coefficients, they cannot be left as local convention changes without an explicit conversion.

   The revision should choose one normalization of the full-space Green kernel and one normalization of the image propagator, state their fluxes, and redo Theorem 7.1 line by line. Every use in Sections 3–6 should then be checked against that convention. Theorem 7.2 and the proposed higher-corner formulae should be rechecked independently rather than inferred from the present normalization. The theorem should also state the analytic hypotheses under which the heat-kernel semigroup property, integrations by parts, and interchange of improper integrals are valid.

3. **The Yangian coefficient in Section 6.6 is contradicted by the manuscript's appendix.**

   The main text states in equation (6.41) that

   $$\mathcal{J}_1 = \frac{a}{2\pi\sqrt{a^2+|\zeta|^2}}.$$

   The appendix defines the same family of integrals in equation (8.25), including a prefactor $3/(4\pi^6)$ that is absent from the main-text definition. Its final closed form, equation (8.45), is

   $$\mathcal{J}_n = -\frac{a}{2\pi\sqrt{a^2+|\zeta|^2}}\left(1-\frac{\binom{2n}{n}}{4^n}\right)\zeta^{n-1}.$$

For $n = 1$, this equals $-a/(4\pi\sqrt{a^2 + |\zeta|^2})$: it has the opposite sign and half the magnitude of equation (6.41). The main-text integrand also changes the relative sign between the image terms at an earlier stage, while the later definition uses a plus sign. Thus the correction in equations (6.42)–(6.44), including the claimed $+\hbar/(2\pi)$ Yangian coproduct term, does not follow from the calculation provided.

This is a decisive check for one of the paper's headline examples. The author should reconcile the integration measure, image signs, orientation of the fused lines, and normalization of the generators, and then repeat the comparison with the standard Yangian coproduct. The resulting coefficient should also be compatible with the R-matrix normalization in equation (6.31).

Even after that correction, the conclusion should be stated more narrowly. The paper checks a leading rational R-matrix term and one level-one coproduct correction. It does not establish the defining relations, Serre relations, full coproduct, Hopf axioms, or an all-orders equivalence with $Y_\hbar(\mathfrak{g})$. The result is potentially valuable evidence for a Yangian action or boundary realization, especially in relation to the Costello–Witten–Yamazaki construction from four-dimensional Chern–Simons theory, but it is not yet a construction of the full Yangian. The assumptions on $\mathfrak{g}$, the completion, the spectral parameter, and permitted representations should also be stated.

4. **The higher-codimension defect coupling in Section 2.7 appears not to satisfy the classical master equation in the stated generality.**

   For the bulk interaction
   $$S_d = \int \eta_i\, d\phi^i + \frac{1}{2}\Pi^{jk}(\phi)\eta_j\eta_k$$
   and the defect coupling (2.48),
   $$S_{d'd} = \int_D D^i(\theta)\eta_i,$$
   the mixed BV bracket contains, in addition to the total-derivative contribution, a term proportional to
   $$\frac{1}{2}\int_D D^i(\theta)\, \partial_i \Pi^{jk}(\phi)\eta_j\eta_k.$$
   This is generically nonzero. Condition (2.46), involving $\pi^{kl}\partial_l D^i$, constrains the variation generated by the defect Poisson structure but does not by itself cancel the derivative of the bulk Poisson tensor. In geometric terms, one expects an additional compatibility condition saying that the map or vector field encoded by D preserves the relevant Poisson data, possibly together with higher couplings. If D is allowed to depend on bulk coordinates as the surrounding prose suggests, further terms arise.

   The manuscript should compute $\{S_d + S_{d'} + S_{d'd}, S_d + S_{d'} + S_{d'd}\}$ explicitly, with degrees and signs, and formulate the exact target-side condition equivalent to its vanishing. The consequences for the module construction in Section 3.5 should then be revisited. Since extended defects are advertised as a systematic outcome of the framework, this cannot be deferred as an incidental technicality.

5. **The interface identities in Section 2 and the higher-bracket configuration-space argument in Sections 3 and 5 require correction and proof.**

   Equation (2.17), presented as the second nontrivial L_∞-morphism identity for an interface, repeats two dummy-index versions of the same term involving a derivative of Π^k and does not contain the required contribution with the output indices k and l exchanged. Schematically, the last pair should involve terms of the form $F^{il}\partial_i\Pi^k$ and $F^{ki}\partial_i\Pi^l$, with the appropriate graded signs. This equation should be corrected and derived from the BV anomaly computation. More generally, checking the first two orders and declaring the remaining tower analogous does not by itself establish the claimed equivalence with a full L_∞ Poisson morphism.

There is a separate domain error in the purported recovery of all higher brackets in Section 3.2. The contour γ_n in equation (3.13) fixes $x_1 = 0$ and constrains $|x_2 - x_1| = 1$. The subsequent text says that every x_i with $i \geq 2$ is integrated over all of $\mathbb{R}^d$ using equation (3.14), but x_2 lies on the fixed sphere and cannot be integrated that way. At most the unconstrained variables with $i \geq 3$ can be removed by the stated unit-integral argument. Equation (3.13) also switches between x_i and y_i and ends an indexed product with $\mathcal{O}_k$ rather than $\mathcal{O}_i$. In the holomorphic–topological analogue, equation (5.22) alternates between γ_{k+1} and γ_k and contains an undefined multi-index $\boldsymbol{n}_0$.

These formulae are intended to establish exact recovery of the target's higher brackets, so a corrected configuration-space-chain argument is needed. The author should specify the compactification, orientation, collision strata, differential of the chain, and signs, and explain how the factorization or little-disks operation follows. If the present calculation is only a tree-level heuristic, the theorem-level language should be adjusted accordingly.

6. **The all-orders quantization claims exceed what is demonstrated in Sections 3.3 and 5.3.**

   The graph sum in Section 3.3 is explicitly described as formal and potentially divergent, but the discussion then treats it as an existence argument for deformation quantization. The formula does not clearly specify automorphism factors, graph orientations and signs, compactified configuration spaces, or a renormalization prescription. Its power of $\hbar$ is assigned by the number of bulk vertices, whereas ordinary Feynman counting depends on propagators and vertices; for example, a single ternary interaction joined by three contractions need not have the stated power. These points must be fixed before the sum defines an E_d-algebra deformation.

   The statement that formality of the E_{d+1} operad is equivalent to every classical theory satisfying the classical master equation being anomaly-free is too strong. Operadic formality does not remove theory-dependent local BV anomalies or supply a renormalized solution of the quantum master equation for an arbitrary action. Similarly, the assertion in Section 5.3 that every $cP_{d,m}$ algebra is quantizable for $d \geq 1$ requires a theorem with hypotheses. The cited holomorphic–topological nonrenormalization results must be shown to cover the manuscript's infinite higher-derivative interactions, boundary conditions, and potential boundary counterterms. The author should state whether the result is perturbative, formal in $\hbar$, and local near a chosen vacuum, and should distinguish finiteness of bulk diagrams from the absence of boundary anomalies.

   An acceptable revision could either supply the missing analytic and operadic construction or explicitly present the graph formula as a conjectural prescription supported by low-order checks. What is not acceptable is to acknowledge that the integrals are undefined and simultaneously use them as proof of a general quantization theorem.

7. **The quantum-group and bulk-center claims should be separated into proved, formal, and conjectural components.**

   The relation to known literature is potentially illuminating, but the novelty is currently difficult to identify because the manuscript moves between established mathematical quantization theorems and new field-theoretic claims. Quantization of Lie and quasi-Lie bialgebras is already known through Etingof–Kazhdan and Enriquez–Halbout; perturbative Chern–Simons theory and Kontsevich integrals already produce Drinfeld associators; and four-dimensional Chern–Simons theory already realizes Yangian structures in the work of Costello–Witten–Yamazaki. The manuscript's new contribution should be stated as the generalized-sigma-model organization of these phenomena, the particular boundary or corner realization, or a genuinely new computation, rather than as a new existence theorem where one is not proved.

Conjectures 4.1 and 4.2 are reasonable conjectures, but the Introduction and later summaries often describe their conclusions as recovered or realized. The quasi-Hopf discussion following equation (4.23) is also phrased more strongly than its calculation supports. Powers of $\hbar$ are suppressed in the leading associator and R-matrix formulae: a one-vertex, three-propagator associator graph begins at order $\hbar^2$, as is finally acknowledged by writing $\Phi(\hbar t_{12}, \hbar t_{23})$. The text also announces that a configuration integral will be evaluated and later says that it could not be evaluated. These points should be harmonized.

Likewise, Section 3.6 proves the higher-Hochschild-center statement cleanly only for the free model. In the interacting case, it identifies an associated graded object or the first page of a spectral sequence. This does not, without a convergence and comparison argument, identify the full interacting bulk observables with E_d Hochschild cohomology. The holomorphic–topological chiral analogue is asserted rather than derived. A result-status table in the Introduction or a substantial Conclusion would help distinguish theorem, formal expectation, conjecture, and example throughout the 132-page paper.

8. **The Virasoro example in Section 6.2 should be corrected rather than used as evidence for the general formalism.**

   Equations (6.7)–(6.8) appear to interchange the geometric types of the fields μ and T: μ is written as a quadratic differential and T as a vector field, opposite to the standard Virasoro coadjoint or Beltrami-field assignment. More seriously, the central term is displayed as $T\partial^3 T$. In the usual BV formulation associated with the Virasoro cocycle, the quadratic central term is built from the Beltrami or vector-type field, schematically $\mu\partial^3\mu$, up to normalization and integration-by-parts conventions. The author should derive this example from the corrected sesquilinearity and shifted-sign conventions of Section 5.1 and verify the degrees of all fields. Because this is the most familiar nontrivial chiral example, it provides an important consistency check of the proposed $cP_{d,m}$ construction.

9. **The publication claim should focus on the strongest genuinely new results after they are repaired.**

   The manuscript contains enough potentially interesting material for more than one paper, but its present breadth obscures the argument. In my view, the most promising core consists of a precise $cP_{d,m}$ definition, a generalized Poisson sigma model built from it, a theorem identifying the induced boundary operations, and a carefully normalized universal bulk-to-boundary calculation, followed by a limited set of worked examples. The extended-object formalism and the five-dimensional boundary/Yangian calculation could then be compelling applications if their master equations and coefficients are established.

   The supersymmetric-twist survey is useful context, but many identifications reorganize the classification and examples already cited in the manuscript rather than provide new derivations. It should be made clear which equivalences are proved here, which follow directly from earlier work, and which are expectations based on matching field content. Reducing repetition and adding a conclusion that lists the precise new theorems would substantially improve both readability and impact.

Minor and presentational comments

1. Section 2.3 first uses the expected $(1-d)$ shift and then refers in prose to a $(d-1)$-shifted structure before writing a shifted cotangent target. The sign convention should be corrected and checked throughout.

2. Equation (2.54) writes the cubic quasi-Lie-bialgebra term with coefficient $1/3$. The arity-three term in the earlier master action has coefficient $1/3!$, and the usual Chern–Simons convention for the double also suggests $1/6$. If a pairing or symmetrization convention changes this coefficient, it should be stated explicitly.

3. The definition of a Poisson module in Section 2.5 describes an endomorphism as a derivation of an underlying "commutative A-module structure." A module does not itself have a commutative multiplication. The intended connection and Leibniz identities, including degrees, should be written out.

4. In Section 7.5, the differential is first written with $\hbar\Delta_{\mathrm{BV}}$, whereas equation (7.46) drops $\hbar$. The preceding coordinate expression also uses a gradient of I where the Hamiltonian vector field should include contraction with the Poisson tensor. These formulae should be made consistent.

5. The general reflection formula before Conjecture 7.1 appears to omit the powers of $1/2$ used in the one- and two-corner image propagators. This should be included in the normalization audit requested above.

6. The statement that semisimple Chern–Simons theory has no topological boundary conditions requires qualification. The answer depends on the quadratic Lie algebra, real form, and the class of boundary conditions allowed.

7. A formal category of $U_\hbar(\mathfrak{g})$-modules is braided, but it is not automatically a modular tensor category. Modularity generally requires additional choices and restrictions involving level or root-of-unity specialization, integrable objects, and semisimplification.

8. The appendix's quasi-Hopf definition refers to further compatibility conditions without stating them. Either give a complete definition or cite a precise definition and list only the data used here.

9. The final manuscript should receive a careful copyedit. There are repeated index mismatches, missing words, and inconsistent uses of "Poisson sigma model" in the singular and plural. Several displayed equations run into the margins, and the figures generate numerous font warnings. These issues matter in a long, notation-heavy paper because they make it difficult to distinguish typographical errors from mathematical ones.

Recommendation

I recommend **major revision**. The manuscript is ambitious, conceptually rich, and potentially suitable for a journal serving the high-energy theory and mathematical-physics communities. Its proposed unification of generalized Poisson sigma models, boundary factorization algebras, extended objects, and quantum groups is interesting. However, the present version is not ready for publication: the central shifted chiral Poisson definition is inconsistent, the universal propagator normalization is not controlled across sections, the explicit Yangian coefficient conflicts with the appendix, and one of the main defect couplings appears not to satisfy the classical master equation without additional hypotheses.

A publishable revision should, at minimum, correct these foundational points; recompute all dependent numerical coefficients; provide a coherent proof or explicitly conjectural formulation of the higher-bracket and quantization claims; and clearly distinguish new theorems from leading-order checks, conjectures, and syntheses of known results. If these tasks are carried out carefully, the resulting paper could make a useful contribution.