Referee Report

"Monodromy and geometry of heavy-light Virasoro blocks"

arXiv:2607.09500

Summary of the paper

The manuscript studies semiclassical Virasoro conformal blocks in the heavy-light regime and seeks a direct bridge between the monodromy and bulk-geodesic descriptions. Its central proposal is that, after passing to the holographic coordinate $\mathfrak{w} = \psi_1^{(0)}/\psi_2^{(0)}$, the two eigenvectors of the first-order monodromy matrix associated with a light line encode the two boundary endpoints of the corresponding bulk geodesic. The authors first demonstrate this statement for a known vacuum four-point block. They then use the triviality of the product of three holonomies at a cubic bulk vertex to derive the three relations in equation (2.10), their explicit solution (2.11), and the projectively invariant cross-ratio condition (2.12). The latter is claimed to hold vertex by vertex for arbitrary heavy backgrounds and is the main conceptual result of the paper.

For two equal heavy insertions, the manuscript gives an independent construction in the conical-defect geometry with $\mathfrak{w} = z^{\alpha}$. A bulk point is mapped to an elementary configuration on the complex plane, where momentum balance and similar triangles reproduce the cross-ratio relation (3.11) and the additive light action. The four-point identity block and a Type-I five-point identity block provide checks of this geometric construction.

The principal application is the non-vacuum HHLLL five-point block. Four auxiliary endpoints obey the vertex equations (4.5). After a global conformal transformation, the system is reduced to the quartic equation (4.8), and equations (4.10)–(4.11) give the proposed classical action and conformal block in terms of a selected root. Section 4.3 then presents endpoint equations for a general comb network, an additive prescription in terms of vacuum four- and five-point building blocks, and a claimed upper limit on closed analytic solvability: three light fields from the degree of the elimination equation and three heavy fields from the heavy-background monodromy problem.

Assessment and recommendation

The project is interesting and potentially suitable for a journal such as JHEP or PRD. In particular, a general fixed-point interpretation of the monodromy eigenvectors, a universal vertex equation in holographic coordinates, and a controlled full leading-light HHLLL block would all be useful results for the semiclassical conformal-block community. The Euclidean-plane construction is also appealing and could make the geodesic prescription more algorithmic.

In its present form, however, the manuscript does not establish its main claims to publication standard. The eigenvector-endpoint identification is verified only in one known example before being promoted to arbitrary external and internal holonomies. The new five-point formula is not checked against the previously known superlight result or directly against the standard monodromy problem. Moreover, the root analysis following equation (4.9) contains a logical contradiction, and its assertion that all four roots of (4.8) are real is false in the stated physical domain. There are also concrete formula and indexing errors, and the claimed maximality of the HHHLLL case does not follow from polynomial degree alone.

I therefore recommend **major revision**. I would be willing to reconsider a substantially revised manuscript if the authors supply a general derivation of the key holonomy statement, validate the five-point result independently, correct the errors detailed below, and moderate or prove the computability claims.

Major comments

1. **The eigenvector-endpoint identification requires a general derivation.**

   The conceptual step after equation (2.5) is the identification of the columns of P_i with projective vectors proportional to $(-1, \mathfrak{w}(z_i))^\mathsf{T}$ and $(-1, \mathfrak{w}(y_i))^\mathsf{T}$. The manuscript verifies this by inserting the known vacuum four-point block, computing its accessory parameter, and diagonalizing equation (2.7). This is a useful check, but it is not a proof of the statement subsequently used for a general network.

   The extension is nontrivial in two respects. First, holonomies based at different paths are related by conjugation, so the authors must specify a common base point and the parallel transports implicit in the matrices entering equations (2.8)–(2.9). Second, an internal line corresponds to a composite-channel holonomy rather than the local holonomy of a single boundary puncture. It must be shown that the fixed points of this composite holonomy are exactly the endpoints obtained by analytically continuing the internal geodesic to the boundary, with the correct orientation and ordering. Flatness by itself gives a relation among consistently based holonomies, but it does not by itself establish this geometric interpretation of their eigenvectors.

   This issue is central because equations (2.10)–(2.12), the claimed background independence, and the entire endpoint algorithm rely on it. The authors should give a direct derivation from the flat $SL(2)$ connection or an equivalent worldline/Wilson-line construction. The derivation should state how P_i transforms under a change of fundamental solutions and base paths, and why the cross-ratio relation remains invariant. It should cover both external and internal lines rather than infer the latter from the four-point vacuum example.

   There is also a sign inconsistency in the displayed derivation itself. Below equation (2.3) the Wronskian is defined as $W = \psi_1\psi_2' - \psi_1'\psi_2$, while $\mathfrak{w} = \psi_1/\psi_2$ implies

   $$\mathfrak{w}' = -\frac{W}{\psi_2^2}, \qquad \psi_1\psi_2 = -\frac{\mathfrak{w}W}{\mathfrak{w}'}.$$

   The text immediately before equation (2.14) uses the same identity with the opposite sign. Related signs in the two intermediate identities between equations (2.7) and (2.8) are also inconsistent with the stated definition of W. Since $W/(\psi_1\psi_2)$ enters equation (2.13) and the integration to (2.14), this is not harmless bookkeeping. The final three-point form may be recoverable by changing the Wronskian convention consistently, but the authors must redo equations (2.3), (2.7), and (2.13)–(2.15) with one convention and verify $c_i = -\partial f/\partial z_i$ explicitly.

2. **The domain of validity and the meaning of background independence must be stated precisely.**

   Several assumptions used later are not collected or justified. The explicit geometry of Sections 3 and 4 assumes two equal heavy insertions, $\alpha^2 > 0$, a constant-time Euclidean slice, and $\mathfrak{w} = z^\alpha$

with a chosen branch. In particular, equation (3.1) follows from the displayed Lorentzian metric only after imposing the equal-time saddle $p_t = 0$; the conserved time momentum has otherwise been omitted. The force-balance triangles at each cubic vertex require strict triangle inequalities among the three incident dimensions. These inequalities are also what underlie the signs $a > 0$, $c < 0$, and $u > 1$ used after equation (4.9). The relevant ordering of the insertion angles and the signs of the angular momenta must likewise be specified.

The paper should distinguish three levels of generality: the monodromy statement proposed for an arbitrary heavy background; the universal algebraic vertex equation in the $\mathfrak{w}$ coordinate; and the explicit conical-defect calculation with two equal heavy fields. The vertex equation can be independent of the heavy data in $\mathfrak{w}$ space while the map $\mathfrak{w}(z)$, and hence the physical block in the original coordinates, remains strongly background dependent. This qualification is important for the claims in the abstract and conclusion.

The authors should also explain whether the construction extends to $\alpha^2 < 0$ (BTZ kinematics), generic complex insertion positions, or non-Euclidean orderings by analytic continuation. In Section 3.2, equation (3.5) replaces the signed angular momentum by an absolute value, but the subsequent momentum-balance equation uses oriented phases $e^{i\mu_i}$. The lost sign and the branch selection leading to equations (3.7)–(3.8) need to be restored explicitly.

Finally, the manuscript calls the HHLLL answer "full," although the calculation remains at first order in the light stress tensor, with $\epsilon_l = O(c^{-1})$. The intended statement appears to be that the result is nonperturbative in ratios of the light and exchanged dimensions at leading order in the light expansion, in contrast with the superlight hierarchy. This should be said unambiguously.

3. **The claimed independent Euclidean-geometry check is not algebraically correct as written.**

   The construction around equations (3.5)–(3.8) is meant to provide an independent validation of the monodromy result, but two displayed steps are inconsistent with the definitions of the plane points. With $U = e^{i\theta}\cot(\rho/2)$ and $V = e^{i\theta}\tan(\rho/2)$, one has

$$\cot\rho = \frac{OU - OV}{2}.$$

   Equation (3.5) gives the magnitude of s as the chord product $ZT\,ZT'/(ZU\,ZV)$. Consequently the expression for $s\cot\rho$ in the paragraph before equation (3.7) should involve $OU - OV$, not $OU^2 - OV^2$ as printed. These expressions differ by the nonconstant factor $OU + OV$.

   The sine-law chain in the next paragraph is also incorrect. The manuscript states

$$\frac{ZU}{\sin\varphi} = \frac{ZV}{\sin\gamma},$$

   whereas the triangles and the authors' angle assignments give

$$\frac{ZU}{\sin\varphi} = \frac{OZ}{\sin\gamma} = \frac{1}{\sin\gamma}.$$

   The subsequent algebra appears to use corrected relations without acknowledging the change. In addition, equation (3.4) writes an ordinary endpoint evaluation of a phase factor; integrating $d\theta$ requires the quotient of its values at x_2 and x_1, not their difference. Only the quotient has unit modulus and can equal $e^{i\Delta\theta}$. Together with the missing factor of s in the differential preceding equation (3.4) and the unsigned momentum in (3.5), these errors mean that equations (3.7)–(3.8) and hence the cross-ratio check (3.11) have not been derived by the displayed argument. The authors should reconstruct this section from the definitions, with oriented angles and signed momenta, and demonstrate each integration, similarity, and sine-law identity. A corrected derivation is necessary if Section 3 is to serve as the advertised independent check rather than an illustration of the already assumed result.

4. **The quartic-root claim is incorrect, and the physical branch proof must be replaced.**

   The discussion following equation (4.9) first states that Q_1 has one negative root x_1 and one positive root x_2. A few lines later, the inequality for f'/f is justified using the contradictory statement $x_{1,2} < 0 < u^{-1}$. The displayed monotonicity argument therefore does not follow as written. It also does not prove the preceding assertion that the quartic $P(x)$ has four real roots.

   Indeed, the four-real-root assertion is false even with ordered angles in the range used by the authors and strict triangle inequalities at both vertices. For example, take

$$(\epsilon_1, \epsilon_2, \epsilon_3, \widetilde{\epsilon}_1, \widetilde{\epsilon}_2) = (1, 1, 1.287633749, 0.938263807, 0.469383198)$$

   and

$$(\arg \mathfrak{w}_1, \arg \mathfrak{w}_2, \arg \mathfrak{w}_3) = (0.661086850, 1.797379854, 2.180538847).$$

   Both triples $(\epsilon_1, \epsilon_2, \widetilde{\epsilon}_1)$ and $(\widetilde{\epsilon}_1, \epsilon_3, \widetilde{\epsilon}_2)$ obey the strict triangle inequalities. The definitions preceding equation (4.8) give $\zeta = -0.256568100$, $|\eta| = 0.243721754$, $u = 2$, and

$$P(x) = 0.125227484 - 1.205116454x + 3.733491603x^2 - 0.903621454x^3 - 0.111617494x^4.$$

   Its roots are approximately

$$-11.17590080, \qquad 2.73739915, \qquad 0.17140213 \pm 0.08540688, i.$$

   Thus only two roots are real. There is still a unique negative root in this example, so the physical prescription may be salvageable, but the stronger claim and its proof are not.

   The revised paper should state all hypotheses on the dimensions and positions, prove exactly the property that is required for the physical solution, and remove the false claim. Existence of a negative root and uniqueness are logically distinct; both should be established. The authors should also explain how the physical root is continued across OPE limits and possible discriminant loci. Since equations (4.10)–(4.11) contain several logarithms and fractional powers, specifying the root alone is insufficient: the branches and phases of every logarithm must be tied to a definite Euclidean configuration and continuation prescription.

5. **The new HHLLL result needs independent and physically meaningful checks.**

   Equations (4.8)–(4.11) are the strongest concrete result of the paper, but the only check presented is the degeneration $\epsilon_3 = 0$ and $\widetilde{\epsilon}_1 = \widetilde{\epsilon}_2$ to the four-point problem. That degeneration lies on a boundary of the second vertex triangle and can generate spurious factors in the elimination polynomial, so the authors should show the reduction explicitly, including which roots are discarded and how the normalization is matched.

   At least the following checks are needed:

- expand the answer in the superlight hierarchy and reproduce the five-point result of Ref. [10] of the manuscript, with an explicit map of conventions;
- take independent OPE/factorization limits, including coalescence of the first two light insertions and decoupling of the third light line, and recover the appropriate lower-point blocks and exchanged dimensions;
- differentiate the proposed on-shell action with respect to the insertion coordinates and verify the accessory parameters, Ward identities, and both channel monodromy eigenvalues;
- compare at one or more generic numerical points with a direct solution of the standard monodromy equations (2.1)–(2.4).

These checks are especially important because the result is implicit in a selected algebraic branch and because equation (4.7) is assembled by adding and subtracting artificially completed geodesics. The paper should demonstrate that the endpoint equations make the action stationary and that differentiating the on-shell action does not leave unwanted derivatives of the auxiliary endpoints. The overall normalization and coordinate-independent constants should also be stated consistently.

6. **Several formula errors must be corrected before the results can be assessed.**

   Equation (4.4), which is presented as a check against earlier four-point results, contains the exponent $-\Delta_2 - \Delta_3 + \widetilde{\Delta}_1$ even though there is no third light external dimension in that problem. The standard three-point exponent pattern indicates that this should be $-\Delta_1 - \Delta_2 + \widetilde{\Delta}_1$. This is not merely a cosmetic issue: equation (4.4) is the benchmark used to validate the endpoint prescription, so the corrected expression should be derived and compared term by term with Refs. [4,15].

   There is also an indexing mismatch in the two additive formulas following equation (4.12). In the first displayed sum, the argument θ_{i+1} is paired with ϵ_i, whereas the final formula pairs it with ϵ_{i+1}. The authors should derive the general action with a diagram and test it explicitly for $n = 2$ and $n = 3$ rather than asking the reader to infer the intended convention.

   In Section 3.1, the unnumbered differential immediately before equation (3.4) is missing a factor of s: from $\dot{\theta} = s \cot^2 \rho$ one obtains $d\theta = \pm s \cos \rho \, d\rho / [\sin^2 \rho \sqrt{1 - s^2 \cot^2 \rho}]$. Equation (3.4) does contain s, but the intermediate derivation should be corrected. The manuscript should be checked systematically for similar convention and transcription errors.

7. **The claimed upper bound on analytic computability is not established.**

   After equation (4.12), the authors state that successive elimination gives a polynomial of degree 2^{n-1} and conclude that, once the degree reaches eight, the equation can no longer be solved analytically. Polynomial degree alone does not imply nonsolvability by radicals. A structured degree-eight family may factor or have a solvable Galois group; conversely, "analytic" can include implicit algebraic, special-function, or isomonodromic descriptions. To claim a rigorous threshold, the authors would need to derive the generic elimination polynomial without extraneous factors and establish its generic Galois group (or another appropriate obstruction). Otherwise the statement should be softened to a limitation of the particular elementary-radical elimination attempted here.

   The heavy-sector part of the same argument also needs qualification. For four or more generic heavy insertions, Ward identities no longer determine all accessory parameters, but this is not equivalent to the absence of analytic or implicit monodromy descriptions, and special configurations can be solvable. In addition, the explicit action construction in Sections 3–4 is carried out for two equal heavy fields, whereas the advertised maximal case has three heavy fields. The authors should show how the endpoint action, its Jacobian terms, and its branch prescription work for a generic three-heavy background before claiming HHHLLL as the largest closed analytic configuration.

8. **Novelty and relation to the cited literature should be sharpened.**

   The manuscript cites the established equivalence between monodromy and geodesic computations, geodesic networks, Steiner-tree formulations, and holographic variables in Refs. [6,9,11,12] and Ref. [15]. It should explain more precisely which part of equation (2.12) is new relative to the force-balance and vertex relations already present in that literature, and which part is a new interpretation or simplification. Likewise, the claim that the full HHLLL block was previously available only in the superlight approximation should be supported by an explicit comparison with Ref. [10] and a current survey of related work. The bibliography appears to stop at 2020 despite strong priority and generality claims.

   The paper already cites a study of four-point blocks with three heavy background operators as Ref. [13], but that work is not used where the three-heavy limit is discussed. A revised introduction should separate known ingredients, new derivations, and new formulas clearly. If the general eigenvector theorem and the five-point checks can be supplied, the resulting advance would be interesting; without them, the present novelty claims are difficult to evaluate.

Minor comments

1. Define the cross-ratio convention before its first use and define "Type-I," the comb-channel orientation, and all internal-index conventions. The auxiliary endpoint notation becomes difficult to follow in equations (4.5) and (4.12).

2. Immediately after equation (2.5), write an explicit equation for P_i. The manuscript currently displays an unassigned matrix and uses $\mathfrak{w}_i(x)$ although the insertion endpoint was denoted z_i.

3. Explain the passage from the boundary block in equation (2.15), which contains $\sum_i \epsilon_i \ln \mathfrak{w}'_i$, to the bulk angular action (2.17), which contains $\sum_i \epsilon_i \ln \mathfrak{w}_i$. The Jacobian and cutoff terms are essential when the heavy background is not simply $\mathfrak{w} = z^\alpha$.

4. In Section 2.1 the notation alternates between N and n for the number of insertions. The statement that the Ward identities leave $n - 3$ accessory parameters should use a single convention.

5. The final relation in equation (4.5) is called "somewhat formal." It should instead be derived as a well-defined limiting cross ratio or replaced by the equivalent antipodal-endpoint equation.

6. State the normalization of the monodromy matrices consistently. In the paragraph leading to equation (4.13), the displayed one-light accessory parameter omits the overall factor ϵ_1 implied by the preceding f_1. Equation (4.13) also omits factors present in equations (2.4)–(2.5); if only eigenvectors matter, say which overall factor has been suppressed.

7. The statements "it is clear that the desired root is negative" and "it is clear that $e^{i\theta}e^{i\mu}$ must lie within the angle" hide branch choices that later affect the block. Replace them with explicit orientation arguments.

8. Several long displayed equations, notably (4.7), (4.10), (4.11), and (4.14), are difficult to parse. Introducing short invariant building blocks and listing their transformation properties would make the result easier to verify and reuse.

Recommendation

Major revision. The manuscript contains a promising idea and a potentially significant multipoint-block result, but publication requires a general proof of the eigenvector-endpoint correspondence, a corrected physical-branch analysis, independent checks of the HHLLL formula, correction of the explicit formula errors, and a defensible statement of novelty and analytic scope.