Referee Report

Spinning the Large-Charge Bootstrap: Parity-Even Operators

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Recommendation: Major revision. The paper contains a novel and potentially publishable stress-tensor sum-rule observation, but its main advertised Regge-trajectory conclusion is stronger than what is presently proved, and the detailed solution under the additional assumptions omits physical sign/phase branches. In addition, several internal inconsistencies in the appendices prevent the central spectral formulas from being checked reliably. I therefore cannot recommend publication in the present form, although I would be interested in a substantially revised version that resolves the points below.

Summary of the paper

The manuscript studies the first subleading large-charge bootstrap in three-dimensional conformal field theories with a global $U(1)$ symmetry. The external operators are two lowest-dimension large-charge scalars together with two light probes chosen from a charged scalar, the conserved current J^a, and the stress tensor T^{ab}. This produces a coupled system of six correlator sectors, SS, SJ, JJ, ST, JT, and TT. Sections 2 and 3 introduce an H-basis for the spinning tensor structures and give spectral expansions containing the first descendant of the charge-sector ground state and new charge-Q primaries. Section 4 uses crossing parity, assumed contact behavior at the boundary of radial convergence, and macroscopic large-charge scaling to turn the spectral expansions into polynomial moment equations.

The most important observation occurs in Section 5.1. The $TT, \delta\delta$ equation at $n = 1$ gives the nonnegative sum in equation (5.3). Consequently, at every physical spin $\ell \geq 2$, each state with nonzero stress-tensor coupling obeys

$$\omega_{\ell,i}^{;2} = J_\ell^2 = \frac{\ell(\ell+1)}{2},$$

as stated in equation (5.6). This is the standard conformal-superfluid Goldstone dispersion. The manuscript then uses the polynomial continuation to spin one, equations (5.12)–(5.15), to argue that stress-tensor support must exist and promotes the pointwise result to the existence of a Goldstone Regge trajectory. Under further assumptions of non-degeneracy and of nonzero analytically continued energy at spin zero for every non-Goldstone branch, Section 5.2 and Appendix H argue that the Goldstone trajectory is unique and that all other trajectories decouple from both J and T at this order. The remaining freedom is then classified in the scalar channel, with explicit one- and two-trajectory examples.

The use of the universal stress tensor to obtain the positive sum in equation (5.3) is elegant and is the strongest part of the paper. It is a meaningful advance over the scalar bootstrap and the current-probe analysis discussed in the Introduction. If the precise logical content of this result is isolated and the technical derivation is repaired, it should be of interest to the large-charge, conformal-bootstrap, and effective-field-theory communities.

Assessment of correctness and logical structure

At the level of the displayed diagonal TT moment equation, the inference from equation (5.3) to the physical-spin condition (5.6) is convincing, provided that the spectral kernel and the treatment of exceptional conserved structures are correct. Positivity uses both $|\lambda_{T,\ell,i}|^2 \geq 0$ and $\omega_{\ell,i} > 0$; the latter follows from the assumed uniqueness of the lowest charge-sector state and should be stated explicitly.

The difficulty is that several later conclusions do not follow from this pointwise fact. In particular, equality at a discrete set of physical spins is not, by ordinary analyticity alone, an identity for one fixed analytic Regge branch. The mixed-channel analysis also replaces invariant relative phases by positive square roots. That step changes the low-spin equations and removes genuine branches of the proposed solution. Finally, the appendices do not presently give a consistent or reproducible derivation of the numerical spectral kernels used to reach equation (5.3). These issues affect the principal theorem and the claimed classification, not merely presentation.

Novelty and significance

The paper's best publication case is quite specific: the stress-tensor channel yields a positive null moment that forces the Goldstone energy for every stress-coupled physical state. This should be foregrounded more sharply as the precise new ingredient beyond the earlier scalar and current analyses. The Introduction would benefit from explaining which TT tensor structure supplies the new positivity and exactly why the current alone did not give the same result.

The broader statement that the general answer is a conformal-superfluid EFT accompanied by additional light fields is not yet established. Decoupling a set of spectral residues from J and T does not by itself construct a local EFT, prove that arbitrary residual scalar data are realizable, or establish a complete set of interactions. The more defensible conclusion is a spectral one: within the stated scaling sector, the additional branches are invisible to the current and stress-tensor correlators at this order. This is already interesting and does not need the stronger EFT interpretation.

Major comments

1. **Equation (5.3) proves a physical-spin statement, not yet one analytic Regge trajectory.**

   For each physical value $w_\ell = \ell(\ell+1)/2$, equations (5.3)–(5.6) imply

   $$T_i(w_\ell) \neq 0 \quad \Longrightarrow \quad x_i(w_\ell) = w_\ell.$$

   The manuscript immediately rewrites this as the assertion that any trajectory coupling to T has $x_i(w) = w$, and subsequently introduces a global set of Goldstone trajectories in equations (5.22)–(5.27). This implication is not automatic. The physical values have no finite accumulation point, so ordinary analyticity in spin does not turn equality on an unbounded discrete set into an analytic identity. Zeros of the OPE coefficient may also exchange the stress-tensor support between branches.

A simple two-branch pattern illustrates the gap. With $w(\ell) = \ell(\ell+1)/2$ and $\epsilon > 0$, consider analytic functions

$$x_1(\ell) = w(\ell) + \epsilon \sin^2\left(\frac{\pi\ell}{2}\right), \qquad T_1(\ell) = \cos^2\left(\frac{\pi\ell}{2}\right),$$
$$x_2(\ell) = w(\ell) + \epsilon \cos^2\left(\frac{\pi\ell}{2}\right), \qquad T_2(\ell) = \sin^2\left(\frac{\pi\ell}{2}\right).$$

At every integer $\ell \geq 2$, exactly one branch has nonzero T-weight and Goldstone energy, so all TT moments equal the expected polynomial powers of w. Nevertheless neither branch is identically Goldstone, and the two physical states are non-degenerate at every spin. Thus the non-degeneracy argument in equations (5.42)–(5.44) does not remove this possibility.

The authors should either prove a suitable interpolation theorem–for example, bounded-degree algebraicity, rationality, a Carlson-type growth condition, or an equivalent consequence of the complete moment system–or weaken the headline claim. What follows without this extra step is already substantial: every stress-coupled physical state has the Goldstone dispersion, and the nonzero polynomial fixed by equations (5.13)–(5.15) implies such support at all but finitely many physical spins. The same interpolation issue reappears when Appendix H promotes physical-spin decoupling to the branch identity $V_i(w) = 0$ and when equations (5.40)–(5.41) assign exceptional-spin states to analytically continued roots.

2. **The mixed OPE products in equation (5.9) have unjustifiably lost their relative signs and phases.**

   The diagonal definitions in equation (5.8) determine S_i, V_i, and T_i from absolute squares. They determine only the magnitudes of $\lambda_S^*\lambda_J$, $\lambda_S^*\lambda_T$, and $\lambda_J^*\lambda_T$. Rephasing the exchanged primary changes all of its couplings together and leaves these mixed products invariant, so their relative signs or phases cannot in general be chosen positive. Equation (D.7) correctly retains $\lambda_A^*\lambda_B$, whereas equation (5.9) replaces each such product by a positive square root without a reality or alignment theorem.

   This is not a cosmetic convention. It removes an explicit solution branch. For real spin-one amplitudes s and j, write $S_{1,1} = s^2$ and $V_{1,1} = j^2$. With the invariant sign retained, the relation represented by equation (5.89) is

$$(1 + sj)^2 = 1 + j^2.$$

   For $j \neq 0$, it has two solutions,

$$s = \frac{-1 + \sqrt{1 + j^2}}{j}, \qquad s = \frac{-1 - \sqrt{1 + j^2}}{j}.$$

   The first gives the weight in equation (5.93),

$$S_{1,1} = \frac{V_{1,1}}{\left(1 + \sqrt{1 + V_{1,1}}\right)^2},$$

   whereas the second gives the omitted branch

$$S_{1,1} = \frac{\left(1 + \sqrt{1 + V_{1,1}}\right)^2}{V_{1,1}}.$$

   In the two-trajectory case with $c^2 + m^2 = 1$, this second branch is compatible with the diagonal equations but changes the sign and magnitude of the SV and ST polynomials relative to equations (5.115)–(5.118). General complex phases give a larger family. The $N = 1$ solution and the solution with the stronger non-degeneracy condition may remain unchanged because their low-spin equations force these weights to vanish, but equations (5.50)–(5.58), (5.84)–(5.96), and the weak-nondegeneracy two-trajectory classification must be redone.

The clean formulation is in terms of positive semidefinite rank-one OPE matrices for each non-degenerate exchanged state, retaining their off-diagonal entries. If degeneracies are allowed at exceptional spin, the corresponding summed Gram matrix should be used. Any additional symmetry that forces all relevant entries to be real and aligned must be stated and proved.

3. **The assumptions used to obtain the polynomial system are not the assumptions advertised in the abstract and Introduction.**

   The five "minimal assumptions" omit several substantive inputs:

   - Section 3.2 assigns the new primaries the same large-Q suppression as the first descendant.
   - Section 4.2 assumes that the discontinuities of all required derivatives have finite distributional limits supported only at coincidence, as used in equations (4.18)–(4.24).
   - Section 4.3 obtains only an upper bound on the coincident singularity from the macroscopic limit and then explicitly assumes saturation after equation (4.34).

   Appendix F demonstrates the first-derivative contact jump only for one scalar Goldstone propagator, equations (F.2)–(F.6). It does not establish the claim for a general CFT, arbitrary derivative order, or any of the spinning H-structures used in the table (4.37)–(4.43). Since the objective is to infer Goldstone behavior from bootstrap consistency, an EFT illustration cannot substitute for stating or deriving the general assumption. Moreover, the propagator in equation (F.2) suppresses the sound-speed anisotropy even though equation (1.3) has $c_s^2 = 1/2$; this does not change contact support, but it does change the distributional normalization and should be treated consistently.

   The authors should distinguish carefully among consequences of Euclidean locality, assumptions about renormalized contact distributions, and assumptions about saturation of a large-charge bound. If only a degree upper bound is needed for the $TT, \delta\delta$ null equation, the central physical-spin theorem should be reformulated using that weaker premise. The full classification should then list every stronger premise it actually uses. Throughout Section 4, "polynomial of degree" should be replaced by "polynomial of degree at most" unless nonzero saturation coefficients have been established.

4. **The F- and H-basis appendices contain incompatible tensor normalizations.**

   Several direct comparisons reveal internal inconsistencies that must be resolved before the spectral sums can be trusted.

   First, the final term in equation (A.9) is $F_{TT,0}\,\delta^{ad}\delta^{bc}$. It is not symmetric under $a \leftrightarrow b$ or $c \leftrightarrow d$, and its double trace is inconsistent with the d^2 denominator in equation (A.17). The intended structure appears to be $F_{TT,0}\,\delta^{ab}\delta^{cd}$.

   Second, the explicit factors in equation (A.9), compared with the level-zero H-definitions in equation (2.25), give

   $$H_{TT,2\delta 3} = \frac{1}{4} F_{TT,2\delta 3}, \qquad H_{TT,\delta\delta} = \frac{1}{2} F_{TT,\delta\delta},$$

   not equations (B.21)–(B.22) as printed.

Third, the descendant tables do not follow from the displayed map. Let $a = 3q/(4Q)$. Equation (A.23) has $F_{ST,23} = ae^{\tau}$ and $F_{ST,22} = ae^{\tau}\cos\theta$, with no displayed $F_{ST,33}$. Equations (B.10)–(B.11) at $d = 3$ then give $H_{ST,3} = ae^{\tau}$ and $H_{ST,0} = 2ae^{\tau}\cos\theta$, whereas equation (B.37) gives $(3/2)ae^{\tau}$ and $3ae^{\tau}\cos\theta$. Likewise, equations (A.24) and (B.15) give $H_{JT,\delta} = e^{\tau}/(2\alpha|Q|^{3/2})$, rather than the value $3e^{\tau}/(4\alpha|Q|^{3/2})$ in equation (B.38). The discrepancy is a factor 3/2 per stress-tensor insertion. The paragraph following equation (B.39) does not specify a convention that reconciles the displayed formulas.

These coefficients feed directly into the isolated spin-one descendant terms and the low-spin matching. The authors should fix the tensor decomposition, state whether every table contains full or channel-normalized coefficients, and recompute the maps. A compact table deriving the rational kernels $K_{AB,\alpha}$ in equation (D.7), or an ancillary symbolic notebook, is essential. Appendix D currently leaves all of these kernels unspecified, including the factors whose square produces the central positive expression in equation (5.3).

5. **Exceptional stress-tensor conservation loci are acknowledged but not treated, including a physical point on the claimed Goldstone branch.**

   Appendix C states that the denominators in equations (C.13)–(C.15) are assumed nonzero and that exceptional cases require separate treatment. The main spectral formulas nevertheless use the generic solution for every $\ell \geq 2$. This is not harmless: for $d = 3$ and the claimed Goldstone energy,

$$\Delta_0 - \Delta_1 = \omega_\ell = \sqrt{\frac{\ell(\ell+1)}{2}},$$

   the factor $\Delta_1 - \Delta_0 + \ell - 2$ in equations (C.14)–(C.15) vanishes at $\ell = 8$, because $\omega_8 = 6 = \ell - 2$. The numerator in equation (C.14) becomes $28\Delta_0$, so the singularity is not a removable 0/0 in that parametrization.

   The conservation system must be solved independently at this and every other allowed exceptional locus. The authors must then show either that the space of conserved structures has the same dimension and the kernels in equations (3.27)–(3.51) are smooth in a nonsingular coupling basis, or that additional structures are absent for a clearly stated reason. In particular, the validity of the $TT, \delta\delta$ sum rule at $\ell = 8$ should be checked explicitly. Equation (C.8) also appears to miss a power: its first stress-tensor structure should carry V_2^2 in order to have spin two at point 2.

6. **The large-Q expansion is not shown to be uniform in the contact limit used to form arbitrarily high moments.**

   Equations (3.5)–(3.6) correctly observe that the macroscopic regime can probe states with $\omega = O(\sqrt{|Q|})$. The spectral sums (3.19)–(3.51) and the derivative moments in equations (4.18)–(4.20), however, effectively retain a Q-independent finite-energy sector and then take $\tau \to 0$. These operations need not commute. A contribution of the form

$$|Q|^{-a} e^{-\sqrt{|Q|}\,|\tau|}$$

   vanishes at fixed nonzero τ, but its mth derivative at $\tau = 0$ scales as $|Q|^{m/2-a}$. Such states can therefore affect precisely the high derivative moments used to derive the algebraic system. Even with finitely many trajectories, the sum over spins and the growing polarization multiplicity can compensate a suppression; it is not enough to refer only to the number of trajectories, as in Section 6.

The authors should provide a uniform spectral-weight bound that justifies exchanging the large-charge, spin sum, and contact limits. Alternatively, the result should be presented explicitly as a formal bootstrap theorem for a closed finite-energy scaling sector, with closure included among the assumptions. The abstract should in any case say that the result concerns first subleading order, finite excitation energy, finitely many trajectories, and parity-even exchange.

7. **The low-spin continuation and the physical interpretation of the additional assumptions need sharper formulation.**

   The generating-function argument in equations (5.38)–(5.41) assumes that the branch energies and residues continue regularly to $w = 0$ and $w = 1$. Root collisions, poles, and cancellations of total residues require care, especially in mixed channels where residues are not positive. Appendix H's recurrence proof robustly constrains the polynomial moment sequence and, by positivity, the non-Goldstone current weight at physical spins. Its generating-function proof additionally expands individual roots and residues near $w = 0$ without stating the needed regularity. The recurrence proof should be made primary, and its exact physical-spin conclusion should be separated from any subsequent interpolation theorem.

   The condition $x_i(0) \neq 0$ in equation (5.62) is also physically strong: it excludes every additional branch that becomes gapless at analytically continued spin zero, including possible extra Goldstone-like modes. It should be described as a physical restriction on the spectrum rather than merely a technical assumption. Similarly, the parity-even truncation should be explained as a closure assumption on the s- and u-channel system, including the circumstances under which parity or external quantum numbers guarantee it. The discussion of conformal solids explains one excluded example but does not yet define positively the class of CFTs covered by the theorem.

Concrete revision requests

A revised manuscript should, at minimum:

1. state a precise theorem whose hypotheses include every large-charge, regularity, parity, and trajectory assumption;
2. either prove a branch-interpolation result or replace the analytic-trajectory claim by the physical-spin statement actually implied by equation (5.3);
3. retain the mixed OPE phases/signs and redo the weak-nondegeneracy and two-trajectory classifications;
4. correct the F/H tensor decomposition and descendant normalizations, and provide a reproducible derivation of the spectral kernels;
5. solve the exceptional conservation systems, in particular the stress-tensor system at $\ell = 8$ on the Goldstone dispersion;
6. justify the contact-distribution and limit-interchange operations, or explicitly delimit the formal scaling sector in which they are assumptions; and
7. verify the corrected equations against the standard conformal-superfluid EFT for several spins and several moment orders in all six correlator sectors.

Minor comments

1. Equation (1.2) should be written as an asymptotic relation, $\Delta_Q \sim \alpha|Q|^{d/(d-1)}$, unless all subleading terms have intentionally been suppressed by definition.

2. Section 2 sets the cylinder radius to $R = 1$, while Section 4.3 later sends $R \to \infty$. The restoration of R and the dimensions of τ, θ, energy density, and charge density should be shown explicitly.

3. The scalar and mixed normalizations in equations (3.7), (3.8), and (3.10) require the leading scalar fusion coefficient to be nonzero. This should be stated where the normalized h-functions are introduced.

4. The notation changes from current labels J and JJ to V, SV, VV, and VT in Section 5. The change should be announced explicitly, and the constants and tilded polynomials in the table (5.37) should be defined before use.

5. The affine form $x_2(w) = c^2 w + m^2$ in equation (5.104) is asserted with little explanation. Please derive its degree bound from the recurrence and polynomial constraints, state the allowed signs/ranges, and distinguish the analytic quantities $S_2(0)$ and $S_2(1)$ from the physical low-spin residues $S_{0,2}$ and $S_{1,2}$.

6. Equation (B.19) drops the JT subscripts on all F-functions. Equation (D.7) should include the trajectory label and define the normalization of every coupling. These are especially important once phases are retained.

7. The positivity discussion following equation (5.3) should mention why $\omega_{\ell,i} > 0$ for the states in the sum and how possible zero-energy degeneracies would be handled.

8. The sentence "we arrive to" in Section 4.2 should read "we arrive at." The sentence beginning "And as a result" in Section 5.2.2 should be joined to the preceding sentence.

9. Given the number of displayed equations and assumptions, section-based equation numbering and a one-page dependency table mapping assumptions to conclusions would make the argument substantially easier to audit.

Recommendation

The positive TT mechanism is novel, physically motivated, and potentially significant. In its present form, however, the paper does not yet establish the advertised existence and uniqueness of an analytic Goldstone Regge trajectory, and the weak-nondegeneracy classification is incomplete because of the missing mixed-channel phases. The tensor-basis and conservation issues also require a full technical recheck. I therefore recommend **major revision**. A corrected manuscript centered on the physical-spin sum rule, with a precise interpolation theorem if available, could become suitable for a high-energy theory journal.