Referee Report

"Bosonization versus the Nielsen–Ninomiya theorem"

arXiv:2607.09935

Summary and overall assessment

The manuscript studies how a two-dimensional bosonic lattice model can realize the long-distance physics of a massless Dirac fermion without introducing microscopic Grassmann variables. The setting is the modified Villain compact scalar at the fermionic self-dual radius. After reviewing the relevant form of the Nielsen–Ninomiya theorem and the spin-sensitive version of Coleman bosonization, the authors combine a vortex insertion on a dual plaquette with a half-electric topological ray to define gauge-invariant lattice representatives of the four continuum Weyl fields. These ray-attached operators are given in equation (43).

The main technical result is an exact separated two-point function, equation (49), expressed through the square-lattice Green function and a path-dependent phase. Its exact diagonal specialization in equation (50) and its large-distance expansion in equation (51) give the expected continuum Weyl propagators in equation (54), including an apparent absence of an order $|x|^{-2}$ correction. The authors then invert this correlator and define a reconstructed chiral kernel through equations (59)–(64). The pole of $\widetilde{S}_{+}(p)$ at the origin, equation (65), becomes the desired single zero of the inverse kernel. On the other hand, a Poincare–Hopf index argument requires compensating zeros of $\widetilde{S}_{+}(p)$; these become poles of the inverse and thereby obstruct locality. Figures 6 and 7 illustrate how the number and positions of the propagator zeros depend on the coincident-point contact term while their total index does not. Finally, equations (71a) and (71b) report a failure of Wick factorization at finite lattice spacing, which the authors interpret as an irrelevant two-body interaction between the composite Weyl operators.

The basic idea is interesting, and the exact two-point calculation is a useful advance. Earlier work cited in the manuscript established the modified Villain model as an ultra-local bosonized lattice realization with exact chiral symmetries and anomalies. The present paper goes beyond that observation by constructing explicit disorder-field representatives, computing their finite-spacing correlator, and displaying the nonlocal inverse kernel that reconciles this realization with Nielsen–Ninomiya. The comparison with SLAC/Stacey fermions is particularly instructive: here the microscopic action remains ultra-local, while the nonlocal object is an inverse correlator of composite operators rather than a microscopic quadratic action. The discussion of contact-dependent propagator zeros also makes a potentially valuable connection with the symmetric-mass-generation literature.

I find the central physical picture plausible and likely correct, and I believe the paper can become suitable for a journal such as PRD or JHEP. In its present form, however, the manuscript does not yet define precisely enough which finite-spacing fermionic theory and which inverse distribution are being used. Several steps essential to the headline conclusions are asserted rather than demonstrated, most notably the branch selection that would make the fermion local, the error estimate underlying equation (51), the treatment of poles in the inverse Fourier transform, and the four-point asymptotics. The numerical claims in Figures 5–7 are also not reproducible from the information supplied.

Recommendation

I recommend **major revision**. The recommendation is not based on a failure of the core bosonization mechanism; rather, the exact two-point result and its relation to Nielsen–Ninomiya appear sufficiently novel and interesting to merit publication once the logical and analytic gaps below are resolved. The revision should either complete the finite-spacing fermionization claimed in the abstract or state consistently that the calculation concerns a chosen branch of line-attached bosonic disorder operators on the infinite plane. It should also make the inverse kernel a well-defined distribution, substantiate the claimed long-distance tail, document the numerical analysis, and supply a checkable derivation of the four-point result.

Assessment of correctness and logical structure

The progression from the Villain action in equation (34), through the gauge-invariant ray operator in equation (43), to the Gaussian calculation of equation (49) is well motivated. The diagonal result (50) provides a useful nontrivial check on the normalization (52), and the resulting continuum limit (54) agrees with the bosonization dictionary reviewed in Section III. The topological core of Section IV.D is also sound in outline: a single pole of the propagator at the physical momentum gives a single zero of its inverse, whereas the compensating zeros required on the momentum torus turn into singularities of that inverse. This is the correct qualitative mechanism for evading the locality hypothesis of the theorem.

The current argument is nevertheless incomplete at precisely the points where the qualitative picture is promoted to a statement about a specific lattice Dirac operator. The correlator in equation (64) is not absolutely summable, and $1/\widetilde{S}_+$ is singular at the propagator zeros. Thus the convolution in equation (60) and the Fourier representation in equation (63) require an explicit regulator and distributional prescription. Moreover, topology fixes the total index of the zeros but not their order. It proves nonanalyticity of the inverse, hence failure of exponential locality, but it does not by itself prove the particular $1/|x-y|$ asymptotic asserted in equations (68) and (69). These issues are repairable, but they must be addressed before the detailed correctness of the reconstructed-kernel claims can be assessed.

There is also a definite contact-term inconsistency in the present formulas. With the coincident values prescribed by equation (52), one has $S_-(0) = 2e^{\gamma_E}(1-i)$, whereas the adjoint relation (53) requires $S_-(0) = -S_+^*(0) = -2e^{\gamma_E}(1-i)$. Thus equation (53), and consequently the stated proof of hermiticity in equation (61), cannot hold for the naive contact term used in Figure 6. The zero-contact prescription does repair this particular mismatch, but the manuscript must choose and justify that prescription before calling the reconstructed operator hermitian.

Major comments

1. **The finite-spacing status of the "fermion operator" must be made precise.**

   Section IV states explicitly that the lattice spin θ-angle is not formulated. Consequently, equation (43) defines a bosonic disorder operator attached to a topological ray, not a local fermionic operator in a fully specified lattice theory with a chosen spin structure. The two-point function is correspondingly double-valued: Figures 2–4 and the discussion around equation (47) select one of two branches by hand. This is sufficient for an interesting infinite-plane calculation, but it is weaker than the abstract's claim to have constructed chiral lattice fermion operators. There is also no isolated ray satisfying $\delta C_x = \delta_x$ on a closed lattice, since the sum of a lattice divergence vanishes; a compensating sink or boundary at infinity is implicit in equation (42).

The authors should take one of two clear routes. They may construct the lattice spin term or fermionization map, explain its finite-volume dependence on spin structure, and show explicitly how it removes the ray and selects the branch. Alternatively, they should consistently qualify the abstract, Introduction, Table I, and Outlook: the objects studied are line-attached bosonic representatives whose selected-branch correlators flow to local Weyl correlators. In the latter route, the paper should also define the equivalence relation on paths, prove the asserted two-class structure, and state precisely which deformations of the path leave equation (49) invariant. This distinction matters for the later claims about on-site chirality and gauging; it is not only a matter of terminology.

2. **The infrared regulator and the derivation of the asymptotic correlator need to be completed.**

   Appendix C.1 informally replaces the infinite square lattice by the large-size limit of S^2, uses simple connectedness to gauge-fix n, and then cancels ratios of massless Gaussian integrals in equations (C4)–(C14). Neither the finite lattice approximating S^2 nor its boundary conditions are specified, and the scalar zero mode is not removed before the divergent integrals are cancelled and the contours are shifted into the complex plane. Please give a definite finite-volume or infrared prescription, fix the zero mode, perform the change of variables within that prescription, and then take the thermodynamic limit. This would also make clear why possible winding sectors or boundary terms do not affect equation (49).

   The absence of an order $|x|^{-2}$ term in equation (51) is central: Appendix C.3 uses it to obtain absolutely summable difference fields and to argue that $\widetilde{S}_+$ has no discontinuity away from the origin. Yet equations (C21) and (C26) replace discrete path sums by continuum arc integrals and simply state an $O(r^{-2})$ remainder. The manuscript should derive this error bound, including the effect of the staircase path, the floor functions, and angular sectors near lattice axes. A numerical check alone would not replace the analytic estimate, because the claimed uniqueness of the propagator singularity depends on it.

3. **Define the Fourier transforms and the inverse kernel as distributions.**

   Because $S_+(x) \sim 1/(x_1 + ix_2)$, the sum in equation (64) is only conditionally convergent. The answer can depend on how the infinite lattice is exhausted unless a symmetric summation, finite-volume limit, or equivalent distributional definition is fixed. Appendix C.3 writes the transform of the $1/(x_1 + ix_2)$ tail directly in terms of a Weierstrass zeta function in equation (C35), but provides neither a derivation nor a reference for this identity. Please specify the summation convention, derive or cite (C35), and show that it is the same convention used in the figures.

   There is a second distributional issue after inversion. The compensating zeros of $\widetilde{S}_+$ make $1/\widetilde{S}_+$ singular, so equation (63) requires a prescription at each zero. The authors should state whether a principal value, a finite-volume inverse, or another extension is intended, and verify that the resulting kernel satisfies equation (60). They should also explain whether different allowed extensions add homogeneous or contact terms and whether those additions affect hermiticity, translation invariance, or the claimed long-distance behavior.

4. **Separate the topological nonlocality result from the stronger $1/|x-y|$ claim and clarify contact-term dependence.**

   The pole at the origin in equation (65), together with continuity elsewhere, does support a single physical zero of the inverse kernel and hence the absence of doublers in the reconstructed two-point function. Poincare–Hopf then requires zeros of $\widetilde{S}_+$ with total index $+1$. This is already a clean proof that the inverse cannot be analytic on the momentum torus, and therefore cannot be exponentially local. However, the existence and total index of zeros do not establish that every zero is simple. Without the local form of the zeros, the $O(1/|x-y|)$ statements in equations (68) and (69) do not follow. The authors should either prove simplicity and calculate the local residues, or weaken these equations to the nonlocality statement that is actually established topologically.

   The contact-term discussion also needs a physical definition. For the operator in equation (43), the manuscript first obtains $S_\pm(0) = Z_\pm \overline{Z}_\pm$ and later replaces it by zero to enforce the geometric relations (55). Changing a coincident correlator may be a legitimate source counterterm, but it is not automatically an ordinary redefinition of a fixed microscopic operator. More sharply, equations (52) and (53) conflict at $x = 0$: equation (52) gives $S_-(0) = 2e^{\gamma_E}(1-i)$, while equation (53) demands its negative. The hermiticity inference in equation (61) therefore fails for the naive contact term. Setting $S_\pm(0) = 0$ restores the adjoint identity, but then the authors must explain why this is the appropriate definition of the inverse correlator and how it is obtained from the generating functional. Please identify the class of allowed counterterms, state which choice defines the kernel used for each claim and figure, and discuss compatibility with higher-point functions and reflection positivity. Only after this is done is it clear in what sense the locations and number of propagator zeros are nonuniversal operator data.

5. **The numerical zero analysis must be reproducible.**

   Figures 6 and 7 are used to claim that the displayed zeros exhaust the zeros of $\widetilde{S}_+$. For the original coincident value, the single compensating zero is only observed numerically and its coordinates are not reported. For the symmetric contact term, equation (67) pins three zeros, but their simplicity, indices, and the absence of further zero pairs remain numerical statements. The manuscript should specify the position-space cutoff or finite volume, summation shape, treatment of the pole, momentum grid, interpolation and zero-finding algorithm, index calculation, convergence tests, and numerical uncertainties. Figure 5's description as a 100×100 lattice should also be reconciled with the infinite-lattice Green function used in the derivation. Providing the short code or data used for these figures would be the most useful way to make the conclusions checkable.

6. **The four-point result requires a derivation and a more careful physical interpretation.**

   The finite-spacing four-point result is advertised in the Introduction as one of the paper's three main results, yet equations (71a) and (71b) are presented without an exact four-point expression or a derivation of the displayed coefficients. The magnitude mismatch does show that Wick factorization is not exact. It does not determine the leading complex connected correlator or its scaling: a phase correction invisible at the displayed order in the magnitude could change that inference. Please provide the full complex four-point function for the stated geometry, specify its path and branch data, subtract the Wick contractions before taking the asymptotic limit, and derive the leading connected term.

   Moreover, a connected correlator of nonlinear composite operators in a Gaussian bosonic theory demonstrates finite-spacing non-Gaussianity of those operators; by itself it does not establish a two-body interaction or a particular 1PI vertex. If the authors wish to retain the interaction language, they should identify the leading irrelevant operator in an effective action or otherwise show how the connected correlator corresponds to a scattering interaction. Otherwise, "finite-spacing non-Gaussianity, with corrections consistent with an irrelevant deformation" would be the more precise conclusion.

7. **The claims about gauging and symmetric mass generation should be scoped to what is demonstrated.**

   The microscopic bosonic action is ultra-local, so the nonlocal reconstructed kernel need not enter a lattice gauging procedure. This is an important point. However, the paper should say explicitly that the mixed anomaly prevents simultaneously gauging the full chiral symmetry without additional degrees of freedom, and distinguish gauging either anomaly-free factor from gauging both $U(1)_V$ and $U(1)_A$. If the claim is meant to include a local fermionic operator at finite spacing, it also depends on resolving the spin-term and branch issue in comment 1. A short explicit coupling of the Villain variables to one background gauge field would make the locality statement concrete.

   Likewise, the propagator-zero discussion is best presented as an exactly calculable two-dimensional illustration. The symmetric-mass-generation works cited in Section IV.D concern gapped mirror sectors and additional questions of interpolating fields, unitarity, and gauging that are absent here. The present example usefully shows that zero locations depend on contact/operator choices, but it does not by itself settle the broader role of zeros in chiral gauge constructions. The revised literature discussion should make this distinction and state more sharply what was not already obtained in the earlier modified Villain and lattice-fermionization papers cited as Refs. [30]–[35].

Minor comments

1. In Section II, the general index theorem for a section of a rank-n bundle over an n-manifold involves the Euler number $\langle e(V_M), [M]\rangle$, not the Euler characteristic of the total space of V_M. For $V_M = TM$ this becomes the Euler characteristic of M, and for the trivial bundle used later it vanishes, so the application can be repaired without changing the argument.

2. Equation (5) is not correct as written in all dimensions. With the two-dimensional gamma matrices displayed in equation (56), for example, $\det(\gamma^\mu D_\mu) = -(D_1^2 + D_2^2)$. The authors only need the equivalence of zeros, so they can write an absolute-value identity or include the dimension-dependent sign and multiplicity.

3. The assertion after Theorem 1 that adding matrices outside the standard Dirac vector "inevitably destroys all on-site chiral symmetries" is too broad. It should be restricted to the specific standard mass-forbidding chiral structure assumed in condition (B), with remnant or flavored lattice symmetries treated separately. Table I should also make clear that its bosonization entries (A)–(D) describe the reconstructed inverse correlator, not the microscopic bosonic action. In particular, entry (B) depends on a contact prescription that actually satisfies equation (53).

4. Equations (B9) and (B10) do not by themselves determine a unique Green function on the infinite lattice: nonconstant harmonic additions, such as a linear function, can vanish at the origin. Add

the D_8 symmetry and growth conditions, or define G directly by the Fourier prescription in equation (B13).

5. The sentence following equation (54) should compare with equation (28) at $N = 1$, not $N = 2$.

6. The bars on $\overline{Z}_\pm$ suggest complex conjugation, but equation (52) assigns the product $Z_\pm \overline{Z}_\pm$ a complex value. State explicitly that these are independent operator normalizations, if that is the intent, and distinguish this notation from Hermitian conjugation.

7. Equation (C28) defines fields F and H, whereas its label and the label for equation (C32) refer to F and G. The notation should be made consistent.

8. Please correct "not no-site" to "not on-site" in the Introduction, "This symmetry also hold" to "holds" after equation (55), "kernel ... that solve" to "kernel ... that solves" before equation (60), and "a ultra-local" to "an ultra-local" in Section V. The capitalization of "modified Villain" should also be standardized.

9. The appendix titles currently begin with "Appendix:" even though the document class already supplies the appendix letters. Removing the redundant word would improve the section headings.

Conclusion

The exact modified-Villain correlator and the resulting topological explanation of a doubler-free but nonlocal inverse kernel are novel enough to justify a substantial revision. Once the authors precisely delimit the finite-spacing operator being studied, regularize and verify the inverse construction, and support the four-point and numerical claims, the work should make a useful contribution to the lattice-field-theory and bosonization literature.