Referee Report

"Color-factor symmetry using perturbiner methods for tree-level amplitudes of Yang–Mills theory coupled to matter"

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Summary and overall assessment

The manuscript develops a color-dressed perturbiner treatment of tree-level Yang–Mills amplitudes with minimally coupled fundamental complex scalars and Dirac fermions. Its principal objective is to prove directly, using recursive Berends–Giele currents, that every such amplitude containing at least one external gluon is invariant under the color-factor shift associated with that gluon. This invariance implies the fundamental Bern–Carrasco–Johansson (BCJ) relations without first assuming a representation whose kinematic numerators satisfy color–kinematics duality.

Sections 2 and 3 set up the logic. The four-point matter-annihilation amplitude in equation (2.1) illustrates the parallel color and kinematic Jacobi identities, equations (2.4) and (2.7), and the resulting BCJ relation (2.10). Equation (2.13) then records the all-multiplicity fundamental BCJ relation. The color-factor shift is defined in equation (3.2), and equations (3.4)–(3.5) show how its action on a proper color decomposition reproduces (2.13).

Section 4 constructs the recursive machinery. Starting from the minimally coupled Lagrangian (4.1), the authors impose Lorenz gauge and introduce the auxiliary fields in equation (4.7), so that the field equations are at most quadratic. They insert the color-dressed perturbiner ansatz (4.16), derive the recursions (4.17)–(4.26), and obtain the gluon-rooted amplitude formula (4.37). Section 5 applies the color-factor shift to the final color tensor in this formula. The variation is reduced to the contraction of the external polarization with the vector S^μ in equation (5.3). After substituting the recursions, the authors separate the known pure-Yang–Mills contribution from ten matter contributions. The order-g term and one order-g^2 term become longitudinal, equations (5.20) and (5.23); the remaining order-g^2 terms are claimed to cancel by the Lie-algebra commutator in equation (5.27); and the order-g^3 terms are claimed to cancel by antisymmetry in equation (5.31). Consequently S^μ is proportional to $k_P^\mu = -k_n^\mu$, and gluon transversality proves the desired invariance. Appendix A checks the recursion against the familiar scalar and spinor four-point amplitudes.

The overall strategy is well motivated, and the four-point calculation gives a useful check on the conventions. A perturbiner proof is also a natural way to expose the recursive origin of color-factor symmetry. I therefore regard the project as potentially publishable in a journal such as JHEP or PRD. However, the sole new all-multiplicity proof is not yet presented in a form that can be verified reliably. The perturbiner algebra needed to make equation (4.16) meaningful is not defined, Fermi statistics for multiple fermion pairs are not treated, and Section 5 contains several concrete index, prefactor, partition, and relabeling errors. These issues go beyond cosmetic editing because the cancellations in Section 5 depend precisely on the unshuffle measure and on permissible exchanges of word blocks.

Recommendation

I recommend **major revision**. The theorem itself is already known, but the perturbiner derivation is a useful and sufficiently distinct methodological contribution if it is made complete and checkable. Before publication, the authors should define the multilinear source algebra and its graded signs, repair and globally recheck the Section 5 derivation, justify the multiparticle Lorenz constraint used in the proof, and test the novel multi-matter sector beyond the one-pair four-point example. The scope and degree of novelty should also be stated more precisely.

Assessment of correctness and logical structure

The early part of the paper is coherent. With the stated conventions, the equations following from (4.1), the use of the auxiliary fields (4.7), and the residue prescription (4.36) have the expected Berends–Giele structure. The explicit calculations in Appendix A reproduce the color factors and kinematic numerators of equations (2.1), (2.5), and (2.6), including their four-point Jacobi identities. These checks support the basic signs and normalizations of the recursion at low multiplicity.

The intended logic of Section 5 is also plausible. Applying the shift only to the root vertex is appropriate because the selected external gluon does not occur in the current labeled by P. Replacing inverse propagators in (5.3) by the recursions naturally generates three- and four-block unshuffle sums. The simplification of the order-g scalar and spinor terms to the longitudinal expression (5.20) can be checked directly. Likewise, the matrix structure displayed just before (5.27) would vanish because $[T^{\mathsf{d}}, T^{\mathsf{a}}] = -f^{\mathsf{adc}}T^{\mathsf{c}}$, and the final matrix polynomial preceding (5.31) is antisymmetric under exchange of d and e.

Nevertheless, these local observations do not yet establish the printed proof. The notation $P = Q \cup R$ is not ordinary set union and is not deconcatenation; Appendix A shows that it denotes all oriented complementary subwords with inherited order. Repeated block relabelings in Section 5 assume that this summation measure is symmetric, while amplitudes with multiple fermion pairs potentially require graded rather than ordinary symmetry. Moreover, several displayed equations are literally ill formed. They appear repairable, but because they sit in the only new part of the argument, the authors should provide a corrected derivation rather than asking the reader to infer the intended formula at each step.

Novelty and significance

The physical conclusion is not new. As the manuscript explains, color-factor symmetry for tree amplitudes with fundamental matter was previously proved by Brown and Naculich using the radiation-vertex expansion, and the associated QCD BCJ relations were also established through BCFW recursion. The present advance is methodological: it extends the 2023 perturbiner proof for pure Yang–Mills theory to minimally coupled scalar and spinor fundamentals and organizes the matter cancellations in the same recursive language used for Berends–Giele currents.

That is a worthwhile, though incremental, contribution. It may provide a more adaptable starting point for studying loop-level color-factor symmetry, matter extensions, or homotopy-algebra formulations. On the other hand, the manuscript does not derive a new BCJ identity, construct new color-dual numerators, produce a new double-copy theory, or extend the symmetry to loop level. Much of Section 4 is standard perturbiner technology, and Appendix A substantially overlaps known four-point calculations. The new result is the matter extension of the recursive proof in Section 5. The paper will have a clearer impact if it distinguishes inherited ingredients from this new step and explains what the recursive formulation makes possible that was difficult in the radiation-vertex proof.

Major comments

1. **The perturbiner ansatz requires a precise multilinear source algebra.**

   Equation (4.16) sums only over words with strictly increasing, nonrepeated letters. Taken literally as an ordinary nonlinear plane-wave expansion, it is not closed under the products in the equations of motion: products of lower currents also generate harmonics with repeated external labels. The standard perturbiner construction avoids these terms by attaching nilpotent bookkeeping variables to the one-particle waves, or equivalently by projecting onto the component multilinear in distinct external sources. Neither device is stated here.

   Please define the object whose coefficients are A_P, Ψ_P, Φ_P, and the auxiliary currents. For example, the authors could introduce source-decorated waves $\eta_i e^{-ik_i \cdot x}$, specify their nilpotency and parity, and define every current as the coefficient of the ordered monomial $\eta_{p_1} \cdots \eta_{p_r}$. The paper should then define $P = Q \cup R$ as an oriented unshuffle (all complementary subwords preserving the inherited order), not as set union. Appendix A's six splittings of $P = 123$ should follow from that definition. The analogous definitions are needed for the three-, four-, and five-block sums in Section 5, including any shuffle multiplicities. This is essential for the relabeling arguments that lead to (5.23) and (5.31).

2. **Multiple fermion pairs require explicit graded-sign bookkeeping.**

   The headline theorem includes arbitrarily many spin-one-half external states, but the manuscript never states whether the Dirac perturbiner coefficients or their source variables are Grassmann odd. With Grassmann sources, extracting the coefficient associated with an increasing word produces Koszul signs for general unshuffles. With commuting nilpotent sources, the recursion instead generates amplitudes for labeled, effectively distinguishable fermion lines, and physical amplitudes with identical fermions must subsequently be antisymmetrized. These are different conventions, and equations (4.17)–(4.26) currently display no graded signs that would distinguish them.

   The authors should choose and state a convention, derive the corresponding unshuffle signs, and explain why every block exchange used in Section 5 is legitimate. If the simpler commuting-source convention is intended, the claim should first be formulated for fixed labeled fermion-line contractions; the authors should then show explicitly that antisymmetrizing identical external fermions preserves the color-factor symmetry. If a fully Grassmann-valued perturbiner is intended, the graded recursions and the cancellations must be written with the relevant signs. Without this step, the four-point example with one fermion line does not establish the result for the multi-fermion amplitudes emphasized in the abstract.

3. **Section 5 needs a complete algebraic audit and a corrected derivation.**

   There are multiple definite problems in the formulas that constitute the new proof:

- In equation (5.4), the pure-gauge term proportional to $-k^{\mu}_{BC}A^{\mathsf{b}}_{\lambda B}$ contains $G^{\lambda\mu\mathsf{c}}_{C}$. This repeats the free index μ and leaves the outer ν uncontracted. Substitution of (4.22) shows that the last index must be ν, namely $G^{\lambda\nu\mathsf{c}}_{C}$.
- Equation (5.7) contains the impossible condition $\sum_{A=D\cup I\cup A}$. The left-hand word should evidently be P, but all word assignments in that step should be checked.
- In the second line of equation (5.10), the term $-\overline{\widetilde{\Psi}}_B\gamma_\nu\Psi_C H^{\nu\mu\mathsf{a}}_{A}$ has lost the common factor $g(T^{\mathsf{a}})^{mn}$ and the sum over $P = A\cup B\cup C$. Later formulas treat this term as though the prefactor and sum were present.
- After equation (5.15), the last two four-block spinor terms are said to cancel under $A \leftrightarrow B$. This cannot be right because B labels the antifermion current, whereas A labels a gauge current. The displayed terms cancel under the simultaneous relabeling $(A,\mathsf{a}) \leftrightarrow (D,\mathsf{d})$.
- An intermediate line in equation (5.18) places the term $2k_A\cdot(k_C - k_B)\widetilde{\Phi}^{*}_{B}\Phi_C$ inside a four-block sum $P = A\cup B\cup C\cup D$, even though it contains no D. The next equality returns it to a three-block sum, suggesting that the intermediate formula is not the intended one.
- In equation (5.28), the term $\widetilde{\Phi}^{m*}_{B}\Xi^{\nu m}_{C}$ has an inconsistent repeated index. It must carry the matter index n on Ξ_C to contract with $(T^{\mathsf{g}})^{mn}$ and to produce the matrix products in the next line.

These errors look largely typographical, and the intended cancellations may survive their correction. Their concentration in the central proof is nevertheless serious. The authors should rederive (5.4) through (5.31) from the corrected graded recursion and check every free index, power of g, word partition, and relabeling. A compact table listing the origin and final fate of each S^{μ}_{i} term, or a machine-readable symbolic check supplied as ancillary material, would make the all-multiplicity identity substantially more convincing.

The transition from the imported pure-Yang–Mills result (5.6) to the single additional contribution (5.8) also deserves a fuller justification. The text currently points to one occurrence of $k^2_I A_I$ in the earlier pure-gauge reduction, but does not demonstrate that this is the only place where the matter terms in (4.17), the auxiliary-current recursion, or the Lorenz constraint can modify that reduction. Please reproduce the relevant short derivation or give an exhaustive structural argument for why S^{μ}_{1} is the complete correction to (5.6).

4. **The multiparticle Lorenz condition used in the proof should be established.**

   Equation (4.27), $k_P \cdot A_P = 0$, is used critically in the passage through equation (5.15). It is stated as the coefficientwise consequence of the Lorenz condition on the full field. That conclusion is immediate only after the independent source monomials underlying (4.16) have been defined. In addition, the recursive equations must be compatible with the gauge constraint once scalar and fermion currents are included.

   Please either prove by induction that equations (4.17)–(4.26), together with the one-letter data, preserve (4.27), or cite and adapt a result that covers the present mixed matter system. It would be useful to show explicitly how the matter current conservation identities following from (4.18)–(4.21) enter this check. The authors should also state the residual-gauge convention for off-shell Berends–Giele currents and explain why the residue formula (4.36) is independent of it.

5. **A check that probes the genuinely new multi-matter sector is needed.**

Appendix A verifies amplitudes with one matter-antimatter pair and two gluons. This is a valuable normalization test, but it does not probe the unshuffle signs or the cancellations involving several matter currents that are central to Section 5. The previously known four-point kinematic Jacobi identity is therefore not a strong test of the new theorem.

I recommend adding at least one five-point example with two matter pairs and one gluon. A mixed scalar-pair/fermion-pair process would exercise both types of current; a two-fermion-pair example with distinct flavors would cleanly test the line assignments, after which the identical-flavor antisymmetrization could be displayed. The amplitude obtained from (4.37) should be compared with ordinary tree diagrams, and its variation under (3.2) should be shown to vanish. Such a check would also expose any missing shuffle multiplicities or graded signs before the general proof is finalized.

6. **The scope, color conventions, and color-basis assumptions need to be stated precisely.**

   Equation (4.1) describes one minimally coupled complex scalar and one Dirac field, with no scalar potential, Yukawa coupling, or other gauge-invariant matter interaction. Thus "an arbitrary number of matter fields" means an arbitrary external multiplicity in this specific minimal theory; it does not mean arbitrary matter species or interactions. The abstract, Introduction, and Conclusion should say this explicitly. If several flavors or unequal masses are included, the flavor indices, allowed vertices, and mass dependence of the recursions should be given. If the proof is only for the Lagrangian (4.1), that restriction is perfectly reasonable but should be maintained throughout.

   The Lie-algebra convention also deserves one sentence. Equation (2.3) writes $[T^{\mathsf{a}}, T^{\mathsf{b}}] = f^{\mathsf{abc}} T^{\mathsf{c}}$ without the usual explicit i, while (4.2) contains $-igf^{\mathsf{abc}}A^{\mathsf{b}}A^{\mathsf{c}}$. Please state whether the generators are Hermitian and whether the factor of i is absorbed into f. This matters when a reader checks the sign cancellation in (5.27).

   Finally, the derivation below (3.5) calls the strings of fundamental generators linearly independent. At fixed finite rank, trace and matrix identities can produce relations among sufficiently long strings. The authors should frame the argument in the formal color space/proper basis used by the MJO decomposition, or state the generic-rank assumption and then explain why the result extends to a chosen finite gauge group.

7. **The manuscript should isolate its new contribution and make the claimed streamlining visible.**

   The recursive setup in Section 4 draws on established gauge-plus-matter perturbiner technology, the pure-gauge part of equation (5.6) is imported from the authors' 2023 work, and the symmetry theorem itself was proved in the 2016 color-factor-symmetry papers. The manuscript cites these sources, but it does not sharply state which recursion formulas or cancellations are new relative to the existing gauge-plus-matter literature. A paragraph at the end of the Introduction should make this division explicit.

   The paper also repeatedly describes the new proof as streamlined, whereas the current Section 5 is a long sequence of term-by-term rearrangements. The argument would be more persuasive if the ten matter contributions were organized by a structural lemma, a current-conservation identity, or at least a cancellation table grouped by coupling order and tensor type. The authors should explain what this organization reveals and how it might facilitate the loop-level extension mentioned in Section 6. This would strengthen the case for publication despite the incremental nature of the result.

Minor comments

1. The linearized adjoint Dirac equation in equation (4.9) is missing its right-hand side $= 0$.

2. Equations (5.1)–(5.2) use $\epsilon_{\mu n}$ through the macro `\de`, whereas the external polarization is denoted by $\varepsilon_{\mu n}$ elsewhere. The notation should be uniform.

3. In the final simplified form of equation (5.15), the last scalar product contains an A^{a} without the word subscript A.

4. Near the end of Section 5, "contracting S^μ_n" should read "contracting S^μ."

5. The two occurrences in Appendix A of $\sum_{k=1}^{4} k_i^\mu = 0$ have mismatched dummy indices; either write $\sum_{i=1}^{4} k_i^\mu = 0$ or avoid the summation index.

6. The simplification of the scalar numerators following equation (A.7) also uses $\varepsilon_4 \cdot k_4 = 0$, not only momentum conservation and $\varepsilon_3 \cdot k_3 = 0$ as stated.

7. The spinor formulas after equation (A.10) switch from m_f to an unadorned m. The mass notation should be consistent.

8. In the scalar four-point discussion, the final specialization should refer consistently to $\widetilde{\phi}_1^*$, which is the quantity appearing in equation (A.7).

9. The sentence introducing equation (4.16) should read "This ansatz consists of an infinite series." Similarly, "the first sum give rise" in Section 5 should be corrected to "the first sum gives rise," and "et. al." should be "et al."

10. The Introduction would benefit from shortening the broad catalog of perturbiner applications and using that space for a more direct comparison with the gauge-plus-matter recursion most closely overlapping Section 4.