Referee Report

Superspace invariants and 3-point correlators in 3d $\boldsymbol{\mathcal{N}=3,4}$ SCFTs

arXiv:2607.10133

Recommendation: Reject in the present form, with encouragement to resubmit after a comprehensive reconstruction.

Summary and overall assessment

The manuscript extends the Park–Osborn superspace construction and the auxiliary-polarization-spinor formalism to three-dimensional theories with $\mathcal{N}=3$ and $\mathcal{N}=4$ supersymmetry. Starting from the familiar bosonic structures P_i, Q_i, S_i and the fermionic structures inherited from the $\mathcal{N}=2$ analysis, the authors contract R-symmetry indices with ϵ^{abc} or ϵ^{abcd}. They thereby propose new families U_{ijk} and $\widetilde{U}_{ijk}$, together with the scalar U' for $\mathcal{N}=4$. Sections 3.2–3.5 aim to give a complete set of generators, relations, inversion parities, and point-exchange properties. Section 4 then uses these objects to list many low-spin three-point structures and to impose current-conservation constraints. The principal physical conclusions are that conserved $\mathcal{N}=3$ correlators contain one parity-even and one parity-odd structure, while their $\mathcal{N}=4$ counterparts contain two parity-even structures and one parity-odd structure, with the second even structure associated with mirror-symmetry breaking.

There is a worthwhile paper within this program. A reliable invariant-ring description for $\mathcal{N}=3,4$ superspace would be a useful technical reference, and the proposed organization of the orientation-tensor sector is natural. The extension to higher Lorentz spin could also provide useful input for future superconformal-block calculations. The manuscript is generally well motivated, and its attempt to connect the $SO(4)$ orientation tensor to the mirror-odd structure known from earlier $\mathcal{N}=4$ supercurrent analyses is physically interesting.

In its present form, however, the paper does not establish the correctness, completeness, or advertised scope of its results. Several foundational transformation rules are inconsistent with the subsequent parity classification. The permutation table contradicts displayed correlators. The conservation law used in Section 4 is not the shortening condition for the $\mathcal{N}=4$ scalar stress-tensor supercurrent, so the comparison with the known $\mathcal{N}=4$ result is not actually performed. There are also concrete wrong-homogeneity monomials and undefined coefficients in the examples used to support the headline counting. Finally, the calculation is not supplied in a form that permits the large invariant reductions and conservation systems to be checked. These are not isolated typographical issues: they affect the central claims and require the invariant and conservation analyses to be rerun.

Major comments

1. **The scope is restricted to R-singlet external operators and to particular assumptions about operator identity.**

   The index-free operators in equation (4.1) carry Lorentz polarization spinors but no external $SO(\mathcal{N})$ indices or R-symmetry polarizations. Consequently, the construction as presented treats R-singlet external superfields. It does not describe a general superconformal primary in an arbitrary R-symmetry representation. In particular, important flavour current and half-BPS sectors of $\mathcal{N}=3,4$ theories fall outside the stated formalism. This restriction should be made explicit in the title, abstract, and conclusions unless external R-symmetry representations are incorporated with the appropriate parallel-transport structures.

There is a separate problem with point exchange. Section 4 repeatedly imposes a $2 \leftrightarrow 3$ symmetry merely because the two entries are both denoted $\mathcal{O}_0$, or imposes full permutation symmetry because the spin labels coincide. Equal spin does not imply identical operator species. For example, the basis preceding equation (4.5) is symmetrized under exchange of the two scalar insertions, while equation (4.6) subsequently presents $\Delta_2 = \Delta_3$ as a consequence of conservation. If the two scalars were identical, equality of their dimensions was already assumed; if they were distinct, the exchange symmetry used to discard $\widetilde{U}_{100}$ was unjustified. The same issue affects the vanishing claims for $\langle \mathcal{O}_1\mathcal{O}_0\mathcal{O}_0\rangle$ and $\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\rangle$.

The authors should first give unsymmetrized bases for specified operator species and then project onto the correct representation of the permutation group when insertions are genuinely identical, including their Grassmann statistics and any flavour tensors. In general it is the coefficient vector, not every chosen monomial separately, that must be invariant or anti-invariant under a swap. All Section 4 counts should be revisited after this distinction is enforced.

2. **The central supercurrent comparison uses the wrong shortening problem.**

   Equations (4.2)–(4.3) describe a first-order divergence constraint for a symmetric-spin superfield. Even for that class, the displayed formula suppresses the R-symmetry index: the condition should specify $D^{a\alpha}\mathcal{J}_{\alpha\cdots} = 0$ for every a. More importantly, this is not a universal definition of a conserved supercurrent multiplet.

   The distinction is essential here. The cited $\mathcal{N} = 3$ stress-tensor supercurrent is a real spinor J_α of dimension 3/2 satisfying a first-order constraint. By contrast, the cited $\mathcal{N} = 4$ stress-tensor supercurrent is a real scalar J of dimension one satisfying the traceless second-order condition

$$D^{I\alpha} D^K_\alpha J = \frac{1}{4}\delta^{IK} D^{L\alpha} D^L_\alpha J.$$

   The first-order operation in equation (4.3) is vacuous at spin zero, and the $\mathcal{N} = 4$ scalar ansatz (4.37) is never subjected to the second-order condition. Thus the manuscript has not reproduced the known two-structure $\mathcal{N} = 4$ supercurrent result, despite making that comparison in the Introduction and Discussion.

   There is also an internal contradiction in the $\mathcal{N} = 3$ comparison. The fully antisymmetric spin-1/2 ansatz (4.28) reduces under conservation to one parity-even coefficient in equation (4.29). This agrees qualitatively with the single functional form of the known $\mathcal{N} = 3$ supercurrent correlator, but not with the manuscript's repeated statement that the $\mathcal{N} = 3$ supercurrent has one even and one odd structure. The authors must identify each superfield multiplet precisely, impose its actual shortening condition, and separate stress-tensor supercurrents, flavour currents, and higher-spin current multiplets. The headline claims should then be reformulated to match what has truly been proved.

3. **The inversion-parity and permutation data are internally inconsistent.**

   The problem begins in Section 3.1. The normalized covariant π^a_{ij} is printed as transforming with a plus sign, exactly like R^a_i and σ^a_{ij}. Nevertheless, equations (3.5) and (3.6) classify structures with one π, such as $\epsilon RR\pi$ and $\epsilon\Omega R\pi$, as parity odd while the corresponding σ structures are parity even. These statements cannot simultaneously hold. The earlier $\mathcal{N} = 2$ construction cited by the authors has a minus sign in the transformation of π_{ij}; if that is the intended convention, it must be restored and every dependent sign rechecked.

   Equation (3.27) also maps the B family to $-A_{jik}$ rather than to $-B_{jik}$. More substantively, its table places both U_{111} and $\widetilde{U}_{111}$ in a class invariant under every swap. Equation (4.28) then includes

U_{111} in a correlator declared fully antisymmetric. The two terms in that equation therefore belong to different exchange sectors under the printed table. Similarly, applying equations (3.26)–(3.27) to every term of equation (4.30) makes those terms even under $2 \leftrightarrow 3$, although the two identical half-integer insertions require an antisymmetric correlator. This changes bases and coefficient counts, not merely notation.

The authors should give fully indexed definitions, including a fixed ordering of Grassmann covariants, and derive the inversion and all three transposition actions from those definitions. A machine-checkable permutation matrix for each homogeneous sector would remove the present ambiguity.

4. **Completeness and minimality of the invariant ring have not been demonstrated.**

   Section 3.4 asserts that products of parity-odd structures and products of ϵ-structures reduce to the proposed generators, and it uses this to conclude that a parity-odd or ϵ-invariant occurs at most linearly in a correlator monomial. The manuscript then says that the complete relations are in Appendix B. They are not: the long $\mathcal{N} = 3$ product list is absent from the rendered appendix, no corresponding $\mathcal{N} = 4$ product catalogue is given, and the difficult $\lambda_1^2\lambda_2^2$ and $\lambda_1^2\lambda_2\lambda_3$ reductions in Appendix B.2 are asserted without displaying the candidate space or its reduction matrix. Suppressing the R-symmetry indices as "obvious" is particularly risky because their ordering controls the Grassmann signs at issue above.

   There is also a conceptual omission in the comparison with $\mathcal{N} = 2$. $SO(2)$ has an invariant antisymmetric tensor ϵ^{ab}, so the repeated statement that such a tensor is absent for $\mathcal{N} \leq 2$ or originates from non-abelian R-symmetry is false as written. If every $\mathcal{N} = 2$ ϵ contraction vanishes or reduces to $\{P_i, Q_i, S_i, \bar{R}_i, R', T'\}$, that fact should be shown explicitly.

   A publishable completeness claim needs an independent counting argument, for example a Hilbert series or representation-theoretic multiplicity count, together with a complete reduction system. At minimum, the symbolic code, Grassmann conventions, generic kinematic specialization, reduction matrices, and tests for linear independence should be supplied as ancillary material. The paper should also explain whether "minimal" means a minimal generating set of the ring, a basis in each multidegree, or a basis only after fixing operator statistics.

5. **Several displayed correlator bases are demonstrably invalid or unproved.**

   The most consequential example is equation (4.49), the $\langle \mathcal{O}_2\mathcal{O}_1\mathcal{O}_1 \rangle$ ansatz. Since Q_1 has degree two in λ_1 and U_{200} has degree two, the monomials

   $$Q_1^2 Q_2 Q_3 U_{200}, \qquad Q_1 P_1 P_2 P_3 U_{200}$$

   have degree six in λ_1, whereas a spin-two insertion requires degree four. They therefore cannot occur in this correlator as printed. Yet the coefficient a_{901} of the first invalid monomial is retained as one of the two independent parity-even coefficients in equation (4.50). The flagship two-even-structure conclusion drawn from this example is consequently not well defined.

   There are plausible omitted structures as well. For instance, $P_3 S_3 U_{220}$ has the homogeneity, odd parity, and $1 \leftrightarrow 2$ symmetry appropriate to equation (4.46), while $P_1 S_1 U_{400}$ has the corresponding properties for equation (4.49). Neither is listed. They may be reducible, but no supplied identity proves this. The missing complete product relations therefore obstruct even a basic check of the Section 4 bases.

   The $\mathcal{N} = 4$ discussion also omits correlators with two half-integer-spin external fields. Such operators are not absent in $\mathcal{N} = 4$; an even number of fermionic insertions is compatible with total Grassmann parity, and structures such as U_{110} have precisely the relevant even total λ degree. This is a direct counterexample to the claim that Section 4 enumerates general spinning correlators. The authors should either provide a systematic algorithm and verified basis for every multidegree and operator statistics, or narrow the paper to the finite set of examples actually treated.

6. **The conservation calculation is neither reproducible nor internally stable.**

   Section 4 presents long lists of rational coefficient relations without a sample derivation, a definition of the linear system, or ancillary code. Several entries reveal transcription or algebraic failures. In equation (4.20), $U_{010}Q_1$ appears with both an a_1 and a b_1 coefficient even though the latter is placed in the parity-odd sector; a tilde is evidently missing. Equation (4.25) uses a_{12} twice where the later equations require a_{21}. Equation (4.26) defines δ_3 but uses δ_{13}, and its b_{22} numerator contains a self-cancelling $+7\delta_{13} - 7\delta_{13}$. Equation (4.48) introduces a_{91} and a_{92}, which do not occur in the preceding ansatz (4.46).

   There are more serious logical discontinuities. Equation (4.41) states that conservation of one spin-one operator sets every displayed ϵ-coefficient to zero. Equation (4.42) then allows a second parity-even ϵ sector after imposing an additional conservation condition. A further linear constraint cannot resurrect coefficients already set to zero unless the first solution was obtained by dividing by a quantity that vanishes on an exceptional locus. No such rank jump is analyzed. The prose says $\Delta_3 = 0$, while the printed numerical coefficients actually match $\Delta_3 = 1$ (for example $a_{12} = 9/16$ and $a_{13} = 75/512$). A dimension-zero scalar primary in a unitary three-dimensional theory would in any case be the identity, reducing the three-point function to a two-point function rather than producing two even and one odd structures. The same phenomenon recurs between equations (4.47) and (4.48), where coefficients set to zero under one conservation condition become nonzero under two.

   All conservation matrices should be regenerated from corrected bases and made available. Their ranks must be evaluated separately at every value of dimension differences where the generic solution divides by a vanishing factor. Simple residual checks should be reported after substituting every claimed solution back into the unsolved Ward identities. Without these steps, the numerical coefficient tables and universal structure counts are not trustworthy.

7. **The all-spin conclusions and triangle-inequality statement are extrapolations, not proved results.**

   Section 4 treats selected low-spin correlators and a few families generated by multiplying a lower-spin ansatz by powers of Q_1. It does not analyze an arbitrary triple (s_1, s_2, s_3), and several prospective cases remain commented out rather than completed. The statement in the Introduction that the parity-odd conserved structure obeys spin triangle inequalities is not followed by a derivation; the triangle rule is only invoked as a consistency remark for a special example. Moreover, some displayed conserved examples have no odd structure or vanish altogether, so the meaning of the advertised "one even plus one odd" rule requires qualification.

   The authors should formulate a precise theorem specifying the operator class, statistics, R representation, spin range, and exceptional cases. A general proof could be based on a multigraded invariant count followed by the rank of the conservation map. If such a proof is not available, the abstract and discussion must be restricted to the explicitly verified examples, and the triangle-inequality claim should be removed.

8. **Novelty and the mirror-symmetry interpretation need a precise dictionary to the existing literature.**

   The earlier supercurrent work cited by the authors already established two independent $\mathcal{N} = 4$ scalar-supercurrent structures and identified the second with failure of invariance under the mirror map. The genuinely new content here would be a correct invariant-ring realization and its extension to higher-spin R-singlet multiplets. The paper should therefore give an explicit change of basis between U' and the known two scalar-supercurrent tensors, rather than treating the two-structure count itself as new.

The action of the outer automorphism exchanging $SU(2)_L$ and $SU(2)_R$ should also be derived on every relevant U structure. The statement that "the free hypermultiplet" is mirror symmetric needs correction or qualification: the mirror-odd coefficient distinguishes left and right hypermultiplets, and a single multiplet is exchanged with its twisted counterpart; a balanced left/right theory is the mirror-invariant example. Explicit free left- and right-hypermultiplet correlators would be an excellent normalization and sign check. More generally, checks under the $\theta \to 0$ conformal limit, restriction to an $\mathcal{N} = 2$ subalgebra, and a free $\mathcal{N} = 3$ model should accompany the revised classification.

Minor and presentational comments

- Equations (2.2) and (2.4) omit the required δ^{ab}. Their overall normalization should also be reconciled with the derivative in equation (2.3) and with the conventions inherited from the authors' earlier work.
- Equation (2.5) omits the factor of i present in its spinor version (2.6). In equation (2.7), the lower-index θ formula lacks a transformation arrow and its sign should be checked against the stated raising/lowering convention.
- At $\theta = 0$, the matrix identity $X_i^{-1} - X_j^{-1} = -X_i^{-1}(X_i - X_j)X_j^{-1}$ shows that the transformations in equation (2.16) require overall minus signs. The final left-hand side in that equation should additionally be X_{ij-}^{-1}.
- The definition of Ψ_i^{ab} in Section 3.1 repeats the index a and contains no free b index. The object $\mathfrak{X}_{i+}$ used there should also be defined explicitly.
- The invariant summary (3.8) should display the separate $\mathcal{N} = 4$ generator U' explicitly. Appendix B.2 calls the odd $\lambda_1^2\lambda_2^2$ object U_{220} when it should be $\widetilde{U}_{220}$.
- Equation (4.7) has $|x_{31}|$ rather than the supersymmetric $|\bar{x}_{31}|$. Under the spin-5/2 heading, equation (4.10) is mislabeled as a spin-3/2 correlator. The general conclusion following equation (4.14) refers to $\mathcal{J}_2$ where $\mathcal{J}_s$ is intended.
- The arbitrary-spin statement preceding equations (4.44)–(4.45) is declared for $s \geq 2$ but includes $Q_1^{s-3}P_3^2U_{400}$, which has a negative power at $s = 2$. Its valid domain and the special cases should be stated separately.
- The group relation is $SO(4) \cong (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$, rather than literally the direct product. The manuscript also needs routine corrections such as "up to," "are classified," and consistent singular/plural use of "structure."

Recommendation

The proposed ϵ-invariant organization is interesting enough to merit a new evaluation after it has been rebuilt and independently checked. The necessary work, however, goes beyond an ordinary revision: the foundational sign conventions, invariant reductions, permutation projections, multidegree bases, conservation matrices, and identification of supercurrent multiplets all require correction or completion. Because the manuscript's main publication claims presently rest on inconsistent bases and on a shortening condition that does not address the $\mathcal{N} = 4$ scalar supercurrent, I cannot recommend publication in its current form. I would encourage a resubmission containing a reproducible algebraic supplement, explicit free-theory and subalgebra checks, the correct multiplet-specific shortening conditions, and claims narrowed to the operator classes actually covered.