Referee Report

D-branes in $\mathrm{AdS}_2 \times H^2 \times H^2$ *under the non-Abelian T-duality*

arXiv:2607.10448

Summary and recommendation

The manuscript studies a classical non-Abelian T-duality of $\mathrm{AdS}_2 \times H^2 \times H^2$ by realizing the three two-dimensional factors through a semi-Abelian Drinfeld double based on $A_2 \oplus A_2 \oplus A_2$. It first presents a formal cancellation of the original metric beta function through two loops, constructs the dual metric and two-form by blockwise inversion, and analyzes one coordinate branch of the resulting geometry. The paper then applies the known Poisson–Lie transformation of a gluing matrix to seven coordinate-aligned boundary conditions and claims several chains among dual D-branes of different dimensions.

The explicit matrix inversion in Section 4, the scalar curvature in equation (4.16), and the canonical gluing-map algebra through equation (5.17) are useful and appear internally consistent. The subject is also potentially relevant to work on non-Abelian duality and open-string boundary conditions. However, the physical interpretation does not follow from these calculations. Most importantly, the dualized algebra is non-unimodular, the full dual dilaton and generalized-supergravity vector are omitted, and the exact dual geometry is not shown to satisfy ordinary beta-function equations. Independently, the central D-brane classification infers dimensions from a determinant, which fixes only a parity and is contradicted by the spectra of the displayed gluing matrices. The claimed chain in equation (5.27) is therefore unsupported.

I recommend **rejection in the present form, with resubmission encouraged only after a complete reanalysis**. This recommendation is not based on matters of presentation: correcting the problems below would change the principal conformal-background and D-brane conclusions, as well as the abstract and conclusions.

Major comments

1. **Non-unimodularity and the missing generalized-supergravity analysis.**

   From equation (4.1), the trace of the structure constants is nonzero:

$$f^b{}_{1b} = f^b{}_{3b} = f^b{}_{5b} = 1.$$

   Thus $A_2 \oplus A_2 \oplus A_2$ is non-unimodular. A non-Abelian duality on such an algebra generically has the familiar Weyl-anomaly term and gives generalized-supergravity equations with a vector I proportional, up to sign and convention, to $\partial_{\tilde{x}_1} + \partial_{\tilde{x}_3} + \partial_{\tilde{x}_5}$. The classical canonical equivalence alone does not establish an ordinary conformal string background. This issue is central here rather than optional: the anticipated vector is not shown to be a trivial or null generalized-supergravity vector.

   The corresponding full determinant-shifted dilaton is also absent. In the normalization used in the manuscript, where the metric beta function contains $\nabla_\mu \nabla_\nu \Phi$, the standard formal shift would give, up to an additive constant,

$$\widetilde{\Phi} = C - \log|\Delta_{12}\Delta_{34}\Delta_{56}| = C - 2\log\sinh r - 2\log\cosh y - 2\log\cosh u$$

on the exterior chart of equation (4.13). Substitution into the exact metric (4.14) does not solve the ordinary equation $R_{\mu\nu} + \nabla_\mu \nabla_\nu \widetilde{\Phi} = 0$. In the coordinate orders (r,t), (y,x), and (u,v), the three residual blocks are respectively

$$\operatorname{diag}(-1, -\operatorname{csch}^2 r), \qquad \operatorname{diag}(-1, +\operatorname{sech}^2 y), \qquad \operatorname{diag}(-1, +\operatorname{sech}^2 u),$$

and

$$\nabla^2 \widetilde{\Phi} - (\nabla \widetilde{\Phi})^2 = -\frac{2}{l^2} - \frac{4}{k}.$$

These nonzero residuals are exactly the sort of terms for which the generalized vector matters. The authors should derive the dilaton and I in a fixed convention and verify the complete generalized equations on every nonsingular coordinate component. Modern work on the non-unimodular NATD anomaly, for example arXiv:1803.07391 and subsequent Poisson–Lie literature, should be discussed. Until this is done, the paper should not describe the complete dual as an ordinary conformal string background.

2. **The small-coordinate beta-function argument does not apply to the exact dual geometry.**

   The field $\widetilde{\Phi} = C - 2\log r$ does solve the ordinary leading equations for the separate truncated metric (4.17), regarded as an exact metric consisting of a singular two-dimensional factor and four flat directions. This does not establish the corresponding claim for the dual metric (4.14). Beta functions contain second derivatives, so terms that are subleading in the metric coefficients can make finite leading contributions to the curvature and Hessian. For example, although $\operatorname{sech}^2 y = 1 - y^2 + \cdots$, the exact (y,x) block has

$$R_{(y,x)} = -\frac{2}{k} + \frac{4}{k\cosh^2 y} \longrightarrow \frac{2}{k} \quad (y \longrightarrow 0),$$

   not zero. Dropping the y^2 and u^2 terms therefore removes precisely the information needed to test the beta function. Likewise, the $-2\log\cosh y$ and $-2\log\cosh u$ terms look quadratic near the origin but have nonzero second derivatives there.

   The text following equations (4.17)–(4.18), together with the abstract and Section 6, should be rewritten. At most it establishes that a different leading singular model has a formal solution; it does not show that a small-coordinate region of the exact NATD background solves the ordinary one-loop equations. A valid replacement is an exact check of the full generalized-supergravity system requested above.

3. **A determinant cannot determine the dimension of the dual D-brane.**

   The reasoning in Cases 2–7 repeatedly uses $\det \widetilde{\mathbb{R}} = \pm 1$ to list possible D-brane dimensions. This gives only the parity of the number of -1 eigenvalues, even if all other boundary conditions have already been established. The matrices in the paper make the failure explicit. Define the typical Euclidean block appearing in equations (5.18) and (5.20)–(5.26) by

$$U(z) = \frac{1}{k^2 + z^2} \begin{pmatrix} k^2 - z^2 & -2kz \\ 2kz & k^2 - z^2 \end{pmatrix}.$$

   Its eigenvalues are

$$\frac{k + iz}{k - iz}, \qquad \frac{k - iz}{k + iz},$$

   neither of which is -1 for finite real z and $k > 0$. Consequently, equation (5.20) has one explicit -1 eigenvalue, one explicit $+1$ eigenvalue, and two such U blocks. It therefore has exactly one

Dirichlet direction at generic nonsingular coordinates: within the paper's own zero-worldvolume-flux setup, the dual of the displayed $D0$ is a $D4$, not an unspecified $D0$, $D2$, or $D4$. This is a direct counterexample to the method used to obtain equation (5.27).

The Lorentzian block in equation (5.18) likewise has eigenvalues

$$\frac{l^2 + \widetilde{x}_2}{l^2 - \widetilde{x}_2}, \qquad \frac{l^2 - \widetilde{x}_2}{l^2 + \widetilde{x}_2},$$

rather than signs. The authors must calculate the -1 eigenspaces, their ranks, and their coordinate loci for every case. They must then construct the dual Dirichlet and Neumann projectors, verify metric orthogonality and integrability, and solve for the embeddings. The determinant-based alternatives and equation (5.27) should be removed.

4. **The displayed cases instead indicate a different and patch-dependent pattern.**

Imposing both the -1-eigenspace definition and the Neumann condition of Appendix A on the matrices shown in Section 5 gives a useful check on what a corrected analysis must address. For the seven nested diagonal ansatzes and with no additional worldvolume two-form, the displayed formulas appear to select

case	consistent dual interpretation
1	$D5$ at generic finite nonsingular coordinates
2	$D4$ at generic finite nonsingular coordinates
3	$D3$ only at $\widetilde{x}_2 = 0$
4	$D2$ only at $\widetilde{x}_2 = 0$
5	$D1$ only at $\widetilde{x}_2 = \widetilde{x}_4 = 0$
6	$D0$ only at $\widetilde{x}_2 = \widetilde{x}_4 = 0$
7	$D(-1)$ only at $\widetilde{x}_2 = \widetilde{x}_4 = \widetilde{x}_6 = 0$.

Thus these particular ansatzes suggest the complementary pattern $Dp \mapsto D(4-p)$, including $D(-1) \leftrightarrow D5$ and $D0 \leftrightarrow D4$, rather than the chains asserted in the paper. Case 7 contains an especially clear logical error: the condition $\widetilde{\Pi} = 0$ does not contradict equations (4.8)–(4.9), because the coordinate-dependent Poisson tensor vanishes at $\widetilde{x}_2 = \widetilde{x}_4 = \widetilde{x}_6 = 0$.

There is an important qualification. The chart used in equation (4.13) has $\widetilde{x}_2 = l^2 \cosh r \geq l^2$, whereas all of the special loci for Cases 3–7 require $\widetilde{x}_2 = 0$. They lie in the inner component, separated from the displayed exterior geometry by the singular hypersurface $\Delta_{12} = 0$. If the intended target is only the chart (4.14), Cases 3–7 provide no consistent zero-flux branes at all. If all nonsingular components of (4.11) are intended, they must be introduced and analyzed separately. The paper needs one coherent choice of global target before making any brane classification.

5. **The boundary-condition framework is internally incomplete.**

Appendix A states that tangent vectors are $+1$ eigenvectors of the gluing matrix. With a nonzero pullback of $B + F$, Neumann directions need not have eigenvalue $+1$; only the Dirichlet -1 eigenspace has the universal interpretation used above. Equation (5.18) is already a counterexample inside the manuscript: it is called space-filling even though its generic eigenvalues are not $+1$. The gauge-invariant worldvolume field $B + F$ is never included, although it is essential to the open-string interpretation and can change the Neumann gluing block.

The claim after equation (5.4) that the Lie-frame gluing matrix may be assumed to be a constant Lie-algebra automorphism is also inconsistent with the examples. In Case 1, $\mathbb{R} = -\mathbb{I}$ is not an automorphism of A_2: for $[T_1, T_2] = T_2$,

$$\mathbb{R}[T_1, T_2] = -T_2, \qquad [\mathbb{R}T_1, \mathbb{R}T_2] = T_2.$$

Either this assumption must be removed or every proposed gluing matrix must satisfy it. The metric-preservation condition $\mathbb{R}^T\Omega\mathbb{R} = \Omega$ should be stated and checked, and the inverse in equation (5.5) must be understood only on the Neumann subspace.

Finally, the seven cases are not a classification: they are one nested, coordinate-aligned choice for each nominal dimension. Other coordinate subsets, orientations, signatures, curved embeddings, and worldvolume fluxes are omitted. For example, the nominal $D0$ of Case 2 has its single Neumann direction along the positive-signature first frame direction, so its physical Lorentzian interpretation is unclear. A revised paper should state the restricted ansatz honestly or provide a genuine classification, and it should work out the boundary conditions and embeddings beyond Case 1.

6. **The group action and the dual coordinate analysis are only local.**

   The assertion that the selected A_2 action is free and transitive on the full AdS_2 factor is false. For the generators used in Section 2,

$$\|\xi_+\|^2 = -\frac{l^2(t^2-r^2)^2}{4r^2}, \qquad \xi \wedge \xi_+ = \frac{r(t^2-r^2)}{2}\partial_t \wedge \partial_r.$$

   The second generator becomes null and the action drops rank at $t = \pm r$. The coordinate transformation printed as equation (4.7), $e^{x_1} = (t^2-r^2)/(2r)$, confirms that only the open region $|t| > r$ is covered. The authors should restrict all claims to this orbit or choose a globally free pair, such as the appropriate $\{\xi, \xi_-\}$ realization on the Poincare patch, and redo the duality.

   On the dual side, equation (4.11) has two singular hypersurfaces, $\widetilde{x}_2 = \pm l^2$. The transformation $\widetilde{x}_2 = l^2 \cosh r$ covers only the positive exterior component. It omits the negative exterior and the inner region $|\widetilde{x}_2| < l^2$, where the timelike coordinate is different. Calling $r = 0$ a "point" and drawing global conclusions from this chart are therefore misleading. All connected components, signatures, and possible extensions must be specified before discussing the global dual or its D-branes.

7. **The horizon, asymptotic, and two-form interpretations are overstated.**

   The condition $G_{tt} = 0$ is not a definition of an event horizon. The manuscript first calls $r \to \infty$ an extremal Killing horizon and then denies the existence of a horizon because there is no finite-r zero. Equation (4.20) in fact identifies this end with the AdS_2 Poincare horizon. Whether the singular hypersurface at $r = 0$ is globally naked requires a causal extension or at least an analysis of null geodesics in the complete chosen component, not only inspection of G_{tt}.

   Equations (4.19)–(4.20) establish a local leading metric limit only in the simultaneous corner $r, y, u \to \infty$. They do not show that the complete string background is preserved. The full formal dilaton behaves as $C - 2(r+y+u)$ in that corner rather than tending to the original constant, and other asymptotic directions do not have the same product limit. Equality of a scalar-curvature limit would in any case be insufficient to define asymptotically AdS boundary conditions.

   Moreover, the two-form (4.15) is exact on the displayed $r > 0$ chart:

$$\widetilde{B} = d\left[-l^2 \log(\sinh r)\, dt - k \log(\cosh y)\, dx - k \log(\cosh u)\, dv\right].$$

   It is misleading to say that the local geometry is "supported" by a nontrivial B-field when $H = dB = 0$. For open strings, gauging it away must be accompanied by the corresponding shift of the worldvolume field F, reinforcing the need to formulate Section 5 in terms of $B + F$.

8. **The two-loop result is formal and the string-theory completion is unspecified.**

   The algebra following equations (2.4)–(2.5) is consistent in the displayed scheme: it gives $\alpha' = l^2 = k$ and $\Lambda = 3/(2l^2)$. However, this sets every curvature radius at the string scale, so $\alpha'\mathcal{R} = O(1)$

and higher-loop terms are not suppressed. It is a zero of a two-loop truncation, not controlled evidence for conformal invariance. The claim must be qualified unless an exact conformal-field-theory argument is provided.

The authors must also specify the theory and renormalization scheme. The cosmological/central-charge term is adjusted without identifying a critical completion, spectator CFT, or internal sector. Conversely, the motivation and D-brane terminology suggest type-II string theory, but the metric extracted from an $F(4)$ near-horizon solution is studied without its gauge, scalar, and flux fields. Since the duality includes a timelike direction, a type-II interpretation must address timelike T-duality and the possible type-II* theory, as well as the RR sector. Without such a completion, the defensible scope is a local bosonic sigma-model calculation rather than a supergravity solution.

9. **Novelty and significance need to be reassessed after the corrections.**

   The construction in Section 4 factorizes into three independent 2×2 A_2 dualizations. Section 3 reviews the standard Klimcik–Severa formalism, and equation (5.15) is the gluing-map result of the cited Albertsson–Reid-Edwards work. The identifiable novelty is therefore the particular direct-product application and its D-brane analysis, but the latter is presently incorrect. The paper should compare explicitly with the author's cited work on non-Abelian duals of $\mathrm{AdS}_{d\leq 3}$ and explain what genuinely new mechanism arises in the product geometry. A corrected exact generalized-supergravity solution and a valid open-string boundary analysis could become useful; the present determinant-based case list is not sufficient for publication in a high-energy theory journal.

Minor comments

1. The statement after equation (2.1) that the metric is "flat" is incorrect; the nonzero scalar curvature quoted in the same sentence already disproves it.

2. The statement following equation (2.2) about the Killing-vector norms is incorrect. In particular, ξ_+ is null on $t = \pm r$, while ξ_- has norm $-l^2/r^2$ throughout the Poincare patch.

3. Section 4 lists three Killing vectors of the dual metric and then states that only a timelike isometry is preserved. This is internally contradictory and should be corrected.

4. The coordinate transformation printed as equation (4.7) and the Poisson tensor printed as equation (4.8) share a duplicate internal label. This produces ambiguous cross-references and must be repaired.

5. Use direct-product notation for the groups and $\oplus$ for their Lie algebras consistently; the manuscript alternates among $\otimes$, $\oplus$, and verbal descriptions.

6. The first expression in equation (A.4) contains a malformed $\mathbb{N}$ term. The full appendix should be proofread against the boundary-condition formulas in the cited literature.

7. State explicitly the allowed signs and domains of l^2 and k, as well as the signature on every coordinate component of the dual.

8. The manuscript needs a thorough language and notation revision. Examples include repeated packages and definitions, inconsistent use of "point" for a singular hypersurface, and frequent grammatical errors that obscure the distinction between a metric, a sigma model, and a string background.