Referee Report

on "Certified boundary-magic witness for state-dependent proto-area in a holographic code"

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Summary and overall assessment

The manuscript studies a deliberately minimal four-qubit construction. The undeformed encoder places the logical qubit directly in the matter qubit M and tensors it with an EPR pair on $G_a G_b$, while the channel Choi state adds an EPR pair on RM. The chosen region is $A = \{M, G_a\}$. Seven one-parameter unitary deformations are then compared. The principal "signal" Hamiltonian, $P_{1,M} P_{1,G_a} X_{G_b}$, rotates the complementary bond leg conditionally on both the matter occupation and the in-region bond leg. Equations (3.1)–(3.3) and Appendix A show that this deformation changes the two nonzero Schmidt values across the cut in a logical-state-dependent way, whereas the six selected comparison families have no such response.

The paper next chooses a decoder $A \to \widehat{L}$ by maximizing coherent information within a two-qubit unitary dilation, and compares its entanglement fidelity with an all-CPTP semidefinite program. It defines a recovery-conditioned quantity $S_{\mathrm{PA}}^{\mathrm{var}} = S(\rho_A) - S(\mathcal{R}(\rho_A))$, which retains a state-dependent response. The resource-theoretic part expands the stabilizer Rényi entropy at the product-EPR Choi state. After projecting its quadratic form away from all one- and two-block tangents of the physical partition $M|G_a|G_b$, the authors obtain the positive value $\mathcal{W}_\partial = 1/(4 \ln 2)$ for the signal generator and zero for the six comparisons. Section 5 then makes an important distinction: for the actual CCKLP resource partition $R|M|\{G_a, G_b\}$, the leading-order witness vanishes for every Hermitian deformation at this base state. A two-bond example instead supplies an operator-support residual for a refined geometry split and an engineered crossing of two candidate bond entropies.

The finite-dimensional entropy calculation is clean, and the manuscript is unusually explicit about several limitations. I find the exact spectra in equations (3.1)–(3.3) and (A.2)–(A.3), the stabilizer-Rényi curvature in equation (4.2), the value in equation (4.5), and the stated Gram ranks to be internally consistent. There is therefore a potentially useful worked example here. Nevertheless, the central physical and resource-theoretic interpretation is not yet established. The positive quantity is not the CCKLP resource witness, is not a magic monotone, and at the stabilizer base reduces to a partition-dependent tangent-space residual. In addition, the "holographic code" is an identity matter factor tensored with an EPR bond, so the observed effect is very close to being built into the controlled bond rotation.

There is also a concrete numerical-definition error in equation (3.9): the reported response of $S_{\mathrm{PA}}^{\mathrm{var}}$ is the value on the stated 21-point logical grid, not the continuum maximum-minus-minimum defined in Section 2. The decoder numbers themselves admit a simple analytic derivation, which should replace the opaque numerical presentation and makes this discrepancy transparent.

Recommendation

I recommend **major revision**. The paper could become publishable as a careful finite-code case study if the authors (i) correct the continuum response, (ii) derive the recovery analytically, (iii) justify or substantially temper the proto-area and boundary-magic interpretations, and (iv) make the exact tangent-space claims self-contained and reproducible. For a journal such as

JHEP or PRD, the revision must also sharpen the significance beyond an engineered three-qubit support diagnostic. If no invariant or operational meaning can be given to $\mathcal{W}_\partial$, and no genuinely encoded example can be supplied, the present results would remain below the expected conceptual threshold.

Major comments

1. **The physical status of the recovery-conditioned "proto-area" requires a derivation.**

   Equation (2.6) defines

   $$S_{\rm PA}^{\rm var}(q,A) = S(\rho_A(q)) - S(\mathcal{R}(\rho_A(q)))\,,$$

   but the manuscript does not derive this subtraction from an operator-algebra RT relation, a complementary-recovery identity, or an approximate area operator. Here the logical inputs are pure. Consequently, entropy in the recovered logical qubit measures decoder-induced mixing rather than an independently defined bulk entropy. Subtracting that mixing can increase the peak-to-peak response: equation (3.9) gives a larger response for $S_{\rm PA}^{\rm var}$ than for $S(A)$. This is not by itself inconsistent, but it makes clear that the quantity depends on the decoder and on the locations of two different extrema.

   The authors should state a criterion under which equation (2.6) deserves an area interpretation. At minimum, they should study its variation over near-optimal fixed decoders, explain whether coherent-information degeneracies can change the recovered entropy, and show the full functions $S(A;q)$, $S_{\rm rec}(q)$, and $S_{\rm PA}^{\rm var}(q)$, rather than only their separate ranges. If no decoder-robust statement is available, the quantity should be presented as a decoder-conditioned diagnostic rather than as proto-area data.

2. **The recovery problem is analytically solvable, and the present grid evaluation makes equation (3.9) numerically incorrect as stated.**

   For the signal isometry, the two environment branches are $|\Phi^+\rangle$ and $|B_\epsilon\rangle$ from equation (A.1). The cross operator on G_a,

   $$X = \mathrm{Tr}_{G_b}\big(|\Phi^+\rangle\langle B_\epsilon|\big) = \frac{1}{2}\begin{pmatrix} 1 & -i\sin\epsilon \\ 0 & \cos\epsilon \end{pmatrix},$$

   Its trace norm is $\|X\|_1 = \cos(\epsilon/2)$ in the coupling range used in the paper. A controlled unitary on MG_a, built from the polar decomposition of X, therefore gives a logical dephasing channel with coherence factor $\cos(\epsilon/2)$. It yields

   $$F_e = \frac{1+\cos(\epsilon/2)}{2} = \cos^2(\epsilon/4), \qquad I_c = 1 - h_2\big(\sin^2(\epsilon/4)\big)\,,$$

   where h_2 uses base-two logarithms. At $\epsilon = \pi/8$, these are exactly the values 0.9903926402 and 0.921821 reported in equation (3.8). Thus the main numerical result has a short analytic explanation. The authors should give the corresponding unitary or Kraus operators and prove the relevant optimality statement, rather than relying on four nonconvex starts and an SDP that optimizes a different objective.

   The same channel gives, in nats,

   $$S_{\rm rec}(q) = h\left(\frac{1+\sqrt{1-\sin^2(\epsilon/2)\sin^2(\pi q)}}{2}\right).$$

Combining this with equation (3.2), at $\epsilon = \pi/8$ the continuous minimum of $S_{\mathrm{PA}}^{\mathrm{var}}(q)$ occurs at $q \simeq 0.713369$, and the response defined by $\max_{q\in[0,1]} - \min_{q\in[0,1]}$ is approximately 0.08496844 nats. The manuscript's 0.084924 is, to the quoted precision, the value obtained at the coarse-grid point $q = 0.70$. Appendix E states that the recovery study uses 21 logical points, but equation (2.6) defines a continuum quantity and equation (3.9) does not label the result as a grid estimate. Equation (3.9), Figure 3, the supplement, and the associated claims must be recomputed or explicitly redefined. The discrepancy is small physically, but it matters in a paper that repeatedly uses "exact" and "certified."

3. **$\mathcal{W}_\partial$ has not yet been shown to be a boundary-magic resource witness in the usual sense.**

   At a stabilizer state, equation (4.2) reduces the stabilizer-Rényi Hessian to

$$Q(H) = \frac{4}{\ln 2}\,\mathrm{Var}(H).$$

   The minimization in equation (4.3) is therefore the squared tangent-space distance, in the covariance metric, from the span of one- and two-block Hamiltonians. For the signal generator, equation (C.3) makes the result especially transparent: three terms are pairwise and the remaining $Z_M Z_{G_a} X_{G_b}/4$ term produces $1/(4\ln 2)$. This is a valid tripartite tangent-support diagnostic, but the paper does not establish an operational task, monotonicity property, or invariance principle that makes it a magic resource quantity rather than a partitioned interaction residual.

   This concern is reinforced by Section 5. The partition used in equation (4.3), $M|G_a|G_b$, is not the CCKLP resource partition. Proposition 1 says that the genuine resource-partition tangent quotient vanishes for every Hermitian deformation in the model. The title, abstract, and discussion should not leave the impression that a nonzero CCKLP-type resource magic has been certified. The authors should establish the behavior of $\mathcal{W}_\partial$ under block-local Cliffords, changes of stabilizer frame, and equivalent generator representatives, and explain why its physical partition is preferred. Absent such a result, terminology such as "projected boundary tangent residual" would more accurately describe what has been proved.

4. **The witness is not sufficient for state dependence on the logical family actually analyzed, and the seven deformations are not an exhaustive classification.**

   The supplement states that six of the 18 bare three-body Pauli directions with positive $\mathcal{W}_\partial$ have a flat $\Delta_q S(A)$ on the real meridian. This is an important qualification to the claimed common selection rule, but it is not discussed in the main text. A positive witness detects a Choi-state tangent outside the pairwise span; it does not by itself imply a response on the one-parameter state family in equation (2.3).

   For the particular population-controlled signal, the phase of a general logical state should drop out after tracing M, so it appears possible to prove that the real meridian contains the global entropy extrema over the full Bloch sphere. That proof should be included. More generally, the authors should present the full-Bloch response of the 27 bare three-body Paulis and state precisely which implication, if any, relates it to the 18/9 witness split. The repeated statement that "only" the signal deformation responds must also be explicitly restricted to the seven hand-selected families. A symmetry-based classification or a study of generic linear combinations would substantially strengthen the result and prevent the comparison set from appearing tailored to the desired conclusion.

5. **The claims of exact certification and Proposition 1 need self-contained analytic certificates.**

The ranks 18 and 30, the 18/9 Pauli split, and the value in equation (4.5) are consistent with the Pauli covariance calculation. However, Appendix C gives no explicit minimizer or exact Gram decomposition, and Appendix D proves the central proposition by asserting a complex128 numerical rank of 30. The EPR mirror identity in equations (D.1)–(D.2) is not by itself sufficient, as the authors correctly note, because the mirrored operator need not be Hermitian. The subsequent numerical rank statement does not supply the promised constructive Hermitian proof.

For Proposition 1, the authors should exhibit 30 independent centered Hermitian pairwise actions or an exact nonzero 30×30 minor. For equation (4.5) and the 18/9 split, they should give an analytic stabilizer-coset argument or provide the exact Gram matrix, projection coefficients, and a machine-readable certificate. The distinction between exact symbolic statements and floating-point checks at tolerance 10^{-10} should be maintained throughout. Code and campaign manifests should be deposited with the paper; availability only "upon reasonable request" is inadequate for numerical claims presented as certified.

6. **The model's holographic and coding content is currently too limited for the strength of the interpretation.**

   Equation (2.1) is the identity embedding of the logical qubit into the physical matter qubit, tensored with one EPR pair. The reconstructing region contains that physical matter qubit directly. The signal Hamiltonian is then designed to move the bond conditional on its population. This explains why the exact calculation is tractable, but it also makes the entropy response nearly kinematic. There is no redundant encoding tensor, nontrivial complementary recovery structure at the base point, area operator, or emergent geometric locality.

   The manuscript should explain which precise properties make this more than a quantum-information toy model. A convincing revision would test the proposed witness in a genuinely encoded example—for instance a small stabilizer or approximate code in which the logical matter is not itself a boundary output factor—and determine whether the selection rule survives. Otherwise the holographic claims, keywords, and significance discussion should be narrowed. The relationship to the CCKLP construction and to approximate Bacon–Shor backreaction should be made comparative rather than merely motivational.

7. **The two-bond extension is underdefined and does not establish the resource claim suggested in the abstract.**

   The manuscript never explicitly defines the two "candidate bond entropies" whose lower envelope is plotted in Figure 6, nor does it give their reduced spectra. Assuming they are $S(G_{a,L})$ and $S(G_{a,R})$, their crossing can be derived continuously and occurs at $q \simeq 0.33499509$, rather than merely at the first winning point of a 401-point grid. The exact functions, the crossing equation, and the uncertainty or grid convention should be written in the text. Figure 6 labels the switch as $q \simeq 0.35$, while its caption and Section 5 use 0.335.

   More importantly, the residual $1/\sqrt{2}$ follows immediately by expanding equation (5.3): two of four equal-norm Pauli terms occupy all three blocks of $\{R, M\}|G_L|G_R$. Under the genuine resource partition $R|M|G_{\text{all}}$, the same Hamiltonian is already supported on the two blocks M and G_{all}, so its residual is exactly zero by definition. This single support observation does not show that the two-bond model "retains" the full four-qubit tangent degeneracy for arbitrary deformations. In addition, r_{split} is a Frobenius norm of the operator representative: adding an operator that annihilates the base state can change it without changing the physical first-order state. The authors should either formulate a state-tangent invariant and analyze a nontrivial class of two-bond deformations, or demote this section to an illustrative observation.

8. **The recovery and undefined Clifford benchmarks need clarification.**

   Equations (3.4)–(3.5) select a decoder by a nonconvex coherent-information objective, whereas equation (3.7) certifies a different, linear entanglement-fidelity objective. Agreement of the selected decoder with the fidelity SDP at one coupling is useful, but does not in itself certify the coherent-information optimization. The analytic treatment requested above can resolve this cleanly; the paper should then state which optimum is proved for all ϵ, which is observed only at $\pi/8$, and whether all optimal decoders give the same recovered-entropy curve.

   The "fixed magic-free Clifford scrambler" introduced after equation (3.9) is not defined in the manuscript or supplement. Its unitary, support, channel, and recovery calculation must be specified before the quoted values $F_e = 0.5$, $I_c = 0$, and $\Delta_q S(A) = \ln 2$ can serve as a control. If it is retained, it should be included in the model definitions rather than appearing only as an unexplained numerical contrast.

Minor comments

1. Table 1 writes $\Delta_q S(A) > 0$ without specifying the coupling domain. The response vanishes at $\epsilon = 0$ and whenever $\sin\epsilon = 0$; the table should state the intended interval or say "generically nonzero."

2. Equation (3.9) should not invite the reading $\Delta S_{\mathrm{PA}}^{\mathrm{var}} = \Delta S(A) - \Delta S_{\mathrm{rec}}$. The extrema occur at different logical states. Their locations should be reported explicitly.

3. Figures 1, 3, and 5 place exact zeros at positive logarithmic plotting floors. A separate zero symbol or broken-axis convention would avoid visually representing exact nulls as small positive effects.

4. The labels for the seven families vary between Table 1, the equations, and the figure axes (for example, "signal," "bond-moving," and "tripartite (irreducible)"). A single terminology would make the comparison easier to follow.

5. The isolated sentence about the $[[5, 1, 3]]$ code after equation (2.5) does not enter any argument. It should either be connected to a concrete test or removed.

6. The statement introducing equation (A.2) should retain the normalized conditioning factor and complex conjugation of equation (2.5) when discussing general logical states; their invisibility is special to the real meridian.

7. The notation H_2 in Section 3 denotes a binary entropy evaluated with natural logarithms, while coherent information and stabilizer Rényi entropy are quoted in bits. The paper does state the conventions, but a notation such as h and explicit units in every numerical table would reduce confusion.

8. The reference to *Inflation is Not Magic* after equation (3.9) is not obviously connected to this finite decoder example. The relevance should be explained or the citation omitted.

9. The two-bond value 4.9×10^{-15} should be described as a floating-point residual for an exactly zero support projection, not as an independent near-zero physical result.

10. The paper should report the deformation-strength range used in every scan and clarify periodic/sign conventions outside the neighborhood of $\epsilon = 0$.