Referee Report

The Euler-Heisenberg action for a $U(1) \times U(1)$ dyon quantum electrodynamics

arXiv:2607.10906

Summary of the paper

The manuscript studies a Dirac fermion carrying two Abelian charges in a doubled-potential formulation of dyon electrodynamics. Starting from the $U(1) \times U(1)$ action in Section II, the authors integrate out the fermion with the Schwinger proper-time method. They claim a two-sector Euler–Heisenberg Lagrangian in equation (25), including nonlinear photon–metaphoton interactions. They then derive a Schwinger pair-production rate, nonlinear field equations, constitutive tensors, and plane-wave dispersion relations in a constant background. Sections IV and V culminate in general parallel and perpendicular refractive indices, equations (56) and (57), and a symmetric-charge example in which two hybrid modes are claimed to have parametrically different birefringence. The one-charge limit reproduces the familiar leading QED birefringence coefficient in equation (62).

The question is worthwhile: a controlled nonlinear effective action for a genuinely dyonic theory, together with a clear account of which optical observables distinguish it from ordinary QED or a dark-photon model, could be of interest. The manuscript is also organized around recognizable physical questions, and the recovery of the leading single-sector QED result is a useful check.

The central two-sector derivation is, however, not correct. The displayed Dirac operator depends on only one charge-weighted linear combination of the two field strengths. Its determinant is therefore the ordinary spinor Euler–Heisenberg functional of that combined tensor. Equations (13)–(25) instead add eigenvalues obtained separately in the two sectors; this is not valid for general field tensors and omits essential cross invariants. With the canonical kinetic action in equation (4), an orthogonal field rotation makes the problem exactly ordinary QED plus a free $U(1)$ gauge field. One propagation eigenmode must consequently remain free to all orders. The constitutive tensors and the advertised two-mode birefringence do not satisfy this exact benchmark. In addition, equations (1) and (4) are not equivalent, and there are independent algebraic errors between equations (56), (59), and (60).

Assessment of correctness, novelty, and significance

The correct single-charge limit does not validate the mixed-sector calculation, because every disputed term vanishes in that limit. Repairing the paper requires replacing the effective action and rederiving essentially all results after equation (12). Within the action actually displayed, the repaired result appears to have substantially less novelty than claimed: a single matter charge vector selects one interacting gauge combination, while the orthogonal combination is free. Genuine basis-independent two-sector optics would require additional structure, such as a second matter species with a non-collinear charge vector, a mass or mixing interaction that fixes the gauge basis, or an explicit source and detector sector that makes the distinction between the two potentials physical.

The comparison with existing literature should be reorganized accordingly. Schwinger's calculation already supplies the determinant once the correct combined field is identified. The dyon effective action of Kovalevich et al. addresses a physically different construction and finds discrete-symmetry-odd terms, while the doubled-potential framework cited from Govaerts must be connected carefully to the local quantum field theory used here. The standard literature on bi-charged matter and Abelian kinetic mixing is also directly relevant. At present, the claimed new phenomena are consequences of an incorrect decomposition rather than established predictions of the model.

Recommendation

I recommend rejection in the present form, with the possibility of resubmission after a complete rederivation. The recommendation follows from errors in the defining action, the heat-kernel invariants, and the main birefringence result. These are not matters that can be resolved by clarification or a limited revision; they change the mode content, the effective operators, and the novelty of the work.

Major comments

1. **Equations (1), (2), and (4) do not define the same classical theory.**

   Equation (2) is

   $$\mathcal{F}^{\mu\nu} = F^{\mu\nu} + \widetilde{G}^{\mu\nu}, \qquad \widetilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}.$$

   In Lorentzian four-dimensional spacetime, $\widetilde{G}_{\mu\nu}\widetilde{G}^{\mu\nu} = -G_{\mu\nu}G^{\mu\nu}$. Consequently,

   $$-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}F_{\mu\nu}\widetilde{G}^{\mu\nu}.$$

   The last term is a boundary term for globally defined Abelian potentials, but the sign of the G^2 term is opposite to that in equation (4). Thus equation (1) contains a wrong-sign gauge kinetic term if interpreted literally, whereas equation (4) contains two healthy Maxwell sectors and is a different action. This also conflicts with the assertion that equation (4) is merely a rewriting and with the later appeal to tree-level unitarity.

   The authors must identify one fundamental action and derive all field definitions and symmetries from it. If equation (4) is intended, equation (1) cannot be used as its equivalent starting point, and the status of the "physical" electromagnetic tensor in equation (2) must be explained separately. If equation (1) is intended, the ghost problem must be resolved before a quantum effective action can be discussed.

2. **The fermion determinant uses one combined field tensor.**

   Equation (6) implies

   $$[D_\mu, D_\nu] = i\mathbb{F}_{\mu\nu}, \qquad \mathbb{F}_{\mu\nu} = q_{(1)}F^{(1)}_{\mu\nu} + q_{(2)}F^{(2)}_{\mu\nu}.$$

   The determinant in equations (9)–(12) can therefore depend only on $\mathbb{F}_{\mu\nu}$. Defining

   $$\Phi = \frac{1}{4}\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}, \qquad \Gamma = -\frac{1}{4}\mathbb{F}_{\mu\nu}\widetilde{\mathbb{F}}^{\mu\nu},$$

the correct leading weak-field spinor result is

$$\mathcal{L}_1^{(4)} = \frac{\Phi^2}{90\pi^2 m^4} + \frac{7\Gamma^2}{360\pi^2 m^4}.$$

The two invariants contain cross contractions that are absent from equation (25):

$$\Phi = q_{(1)}^2 \mathcal{F}^{(1)} + q_{(2)}^2 \mathcal{F}^{(2)} + \frac{q_{(1)}q_{(2)}}{2} F_{\mu\nu}^{(1)} F^{(2)\mu\nu},$$
$$\Gamma = q_{(1)}^2 \mathcal{G}^{(1)} + q_{(2)}^2 \mathcal{G}^{(2)} - \frac{q_{(1)}q_{(2)}}{2} F_{\mu\nu}^{(1)} \widetilde{F}^{(2)\mu\nu}.$$

The step in equations (13)–(21) effectively treats the eigenvalues of $q_{(1)}F^{(1)} + q_{(2)}F^{(2)}$ as sums of eigenvalues of the two tensors. Linearity of the field strength does not imply linearity of matrix eigenvalues, matrix logarithms, or heat kernels. Such a step is available only under restrictive simultaneous-diagonalization conditions, which are neither stated nor valid for the general fluctuations used later.

A simple counterexample makes the problem explicit. Take two constant pure magnetic fields of fixed magnitudes and vary the angle between them. Every separate-sector invariant used in equation (25) is unchanged, while the exact determinant depends on

$$\left| q_{(1)}\vec{B}_{(1)} + q_{(2)}\vec{B}_{(2)} \right|^4.$$

For antiparallel fields the combined tensor can even vanish while equation (25) remains nonzero. The fractional powers in equation (25) and the inverse background powers in equation (37) are further warning signs: a local weak-field expansion from a massive fermion loop is analytic and polynomial around zero field. Equation (25) can reproduce some aligned-background values, but it is not the general effective Lagrangian claimed in the paper.

3. **The displayed model is exactly QED plus a free gauge field after a constant field rotation.**

   Taking equation (4) as the intended kinetic action, let

$$Q = \sqrt{q_{(1)}^2 + q_{(2)}^2}, \qquad C_\mu = \frac{q_{(1)}A_\mu^{(1)} + q_{(2)}A_\mu^{(2)}}{Q}, \qquad D_\mu = \frac{-q_{(2)}A_\mu^{(1)} + q_{(1)}A_\mu^{(2)}}{Q}.$$

   The Maxwell kinetic terms remain canonical, the fermion couples only to C_μ with charge Q, and D_μ is exactly free. All fermion loops are ordinary QED loops with external C fields. The charge-orthogonal field D has $n_\parallel = n_\perp = 1$ for every background and to every order in the loop expansion. In the original basis the interacting and free eigenvectors may look like photon–metaphoton superpositions, but this does not produce two independent nonlinear responses.

   This gives a stringent benchmark for Sections IV and V. For example, when $q_{(1)} = q_{(2)} = q$ and $\vec{B}_{0(1)} = \vec{B}_{0(2)} = B\hat{z}$, set $\alpha = q^2/(4\pi)$. The interacting combination has $Q = \sqrt{2}q$ and background $B_C = \sqrt{2}B$, so ordinary Euler–Heisenberg theory gives

$$\delta n_C = \frac{48\alpha^2}{45m^4}B^2 + O(B^4), \qquad \delta n_D = 0$$

   with the second equality exact. Equation (60) instead gives the leading coefficient

$$-\frac{176\alpha^2}{45m^4}B^2$$

for one branch and a nonzero quartic result for the other. Neither branch has the required mode content.

There is a related renormalization issue. A fermion charged under both $U(1)$ factors produces a two-point function proportional to $q_{(a)}q_{(b)}$, including an $F^{(1)}_{\mu\nu}F^{(2)\mu\nu}$ kinetic counterterm. Subtracting the quadratic heat-kernel term amounts to a renormalization condition, not to the absence of dimension-four kinetic mixing. The condition, its scale dependence, and the running kinetic matrix should be stated. In the C, D basis this is simply the rank-one vacuum-polarization renormalization of C. The claim below equation (57) that mixing arises only through nonlinear operators is therefore misleading.

If other matter, external sources, or detector couplings are intended to make the A/B basis physically preferred, they must be included. With only the action shown, all claimed irreducible two-sector effects are removable by the rotation above.

4. **The constitutive and dispersion formulas fail independent internal algebra checks.**

   Even setting aside the incorrect effective action, equation (59) does not follow from equation (56). In the symmetric limit

$$\beta_{(1)} = \beta_{(2)} = \beta, \qquad \gamma_{(1)} = \gamma_{(2)} = \gamma, \qquad \rho_0^{(1)} = \rho_0^{(2)} = \rho, \qquad B_{0(1)} = B_{0(2)} = B,$$

   equation (56) gives, up to a relabeling of the roots,

$$(n_{\parallel}^{\pm})^2 = 1 + \frac{(\gamma \pm \rho)B^2}{\beta}.$$

   Equation (59) instead has $1 - (\gamma \pm \rho)B^2/\beta$. No exchange of the $+$ and $-$ labels can reverse the common sign of γ. The mixing-suppressed limit in equation (58), which has the plus sign, provides the same immediate check.

   The branch labels are also not paired consistently between polarizations. In the symmetric gauge-sector matrix, the symmetric vector $(1, 1)$ and antisymmetric vector $(1, -1)$ must be followed separately. The $+$ root in equation (56) and the $+$ root in equation (57), as printed, correspond to different gauge eigenvectors. Subtracting them to form δn^+ therefore does not define the birefringence of one propagating mode. Using the coefficients in equation (37) and pairing eigenvectors consistently gives leading coefficients $30\alpha^2B^2/(45m^4)$ and $82\alpha^2B^2/(45m^4)$ within the manuscript's own constitutive system, rather than the -176 and 0 coefficients in equation (60). These alternative numbers still inherit the invalid upstream Lagrangian, but they demonstrate that equation (60) is not even an internal consequence of equations (56) and (57).

   Equation (49) contains a second structural mismatch. From equation (40), the ρ_0 coupling acts in the polarization channel parallel to the background and belongs with the γ diagonal response; the σ_0 coupling acts in the $\vec{k} \times \vec{B}_0$ channel and belongs with λ. Equation (49) pairs ρ with the λ factor and σ with the γ factor. For common backgrounds with $\vec{k} \perp \vec{B}_0$, direct 2×2 determinants in the two transverse polarizations should instead have the schematic forms

$$A_{\gamma 1}A_{\gamma 2} - \omega^4 \rho_0^{(1)}\rho_0^{(2)} B_{0(1)}^2 B_{0(2)}^2, \qquad A_{\lambda 1}A_{\lambda 2} - k^4 \sigma_0^{(1)}\sigma_0^{(2)} B_{0(1)}^2 B_{0(2)}^2.$$

   Finally, the Schur-complement manipulation in equations (41) and (42) is valid only where the chosen diagonal block is invertible. The calculation should retain the full determinant, clear the inverse-block denominators before taking limits, and verify all transverse roots. This is especially important in the decoupling and light-cone limits, where the inverses in equations (44)–(48) are singular. A direct diagonalization in the charge-aligned basis would make the exact free mode manifest and is the simplest check of the result.

5. **The claimed quartic birefringence, critical fields, and dichroism lie outside the controlled approximation.**

   Equation (25) retains the dimension-eight, four-field Euler–Heisenberg operators. Such a truncation determines refractive indices through order B^2/m^4. Expanding rational expressions or square roots built from that truncated action generates apparent B^4/m^8 terms, but dimension-twelve six-field operators omitted from the action contribute at the same order. Hence the quartic coefficients in equations (60) and (62) are not predictions of the calculation. This is particularly decisive for the claimed "protected" negative mode: once its B^2 term is said to cancel, its purported leading B^4 coefficient is entirely sensitive to the omitted operators. The manuscript acknowledges this issue for equation (62) but nevertheless treats the corresponding term in equation (60) as a distinctive signal.

   Likewise, the zeros and poles discussed after equation (54) occur when $C_1 B_0^2$ is of order unity. This is far beyond the weak-field expansion $|qB_0|/m^2 \ll 1$ used to obtain equation (25). These rational-function singularities are truncation artifacts; they cannot establish a physical critical field, an evanescent regime, or the onset of dichroism. Dichroism is an absorptive effect and should be tied to the imaginary part of the appropriate polarization tensor or the full effective action, not inferred from a pole produced by algebraically resumming a real quartic truncation. The discussion should be removed unless it is replaced by a controlled calculation using the full one-loop response.

6. **The backgrounds used in Sections IV and V are not the physical magnetic backgrounds claimed in the phenomenological discussion.**

   Equation (7) defines
   $$\vec{E}_{\text{phys}} = \vec{E}_{(1)} - \vec{B}_{(2)}, \qquad \vec{B}_{\text{phys}} = \vec{B}_{(1)} + \vec{E}_{(2)}.$$
   The choice in equation (33), $\vec{E}_{0(1)} = \vec{E}_{0(2)} = 0$ with both $\vec{B}_{0(1)}$ and $\vec{B}_{0(2)}$ nonzero, therefore gives
   $$\vec{E}_{0,\text{phys}} = -\vec{B}_{0(2)}, \qquad \vec{B}_{0,\text{phys}} = \vec{B}_{0(1)}.$$
   It is not a purely magnetic physical background. In the symmetric example used for equations (59) and (60), the physical background contains parallel electric and magnetic fields of equal magnitude. It cannot be interpreted directly as the magnetic background of a PVLAS-type experiment.

   The authors must specify which sources create $A^{(1)}$ and $A^{(2)}$ backgrounds, which field combination an ordinary laser and detector couple to, and how the calculated eigenmodes are prepared and observed. A physically pure magnetic background must satisfy $\vec{E}_{0(1)} = \vec{B}_{0(2)}$; the simplest choice sets both to zero. Setting $B_{0(2)} = 0$ also exposes the singular ratios in equation (37), which a correct analytic effective action cannot possess. Without a source/detector model, the concluding claim that a deviation from QED could reveal a hidden magnetic charge is not supported.

7. **The pair-production result requires a more restrictive field configuration and correct absolute-value conventions.**

Equations (27)–(29) should be derived from the secular electric and magnetic strengths of the combined tensor $\mathbb{F}_{\mu\nu}$. The assumption $\vec{E}_{(a)} \parallel \vec{B}_{(a)}$ separately in each sector does not ensure that $F^{(1)}$ and $F^{(2)}$ share an eigenbasis; the directions must be mutually compatible. In the pure-electric combined-field limit, the relevant scale is

$$\mathcal{E} = \left| q_{(1)}\vec{E}_{(1)} + q_{(2)}\vec{E}_{(2)} \right|,$$

not a sum of positive field magnitudes. The spinor rate then contains

$$\exp\left(-\frac{n\pi m^2}{\mathcal{E}}\right).$$

This absolute value is essential. Equation (6) defines $q_{(1)} = -e$ and $q_{(2)} = -g$, while equation (13) defines $E_{(a)}$ as a magnitude. With positive e, g, and $E_{(a)}$, the denominator printed in equation (29) is negative, so the displayed exponential grows and the poles in equation (28) lie at negative proper time. Cancellations of the vector combination, including a vanishing pair-production field despite nonzero sector fields, must also be handled.

There is no sharp onset field: the Schwinger rate is nonzero but exponentially small for every nonzero $\mathcal{E}$, with $\mathcal{E} \sim m^2$ only setting a characteristic scale. Equation (29) should be identified explicitly as the pure-effective-electric limit of the more general parallel-field result, and the physical interpretation should be translated through equation (7).

8. **The discrete-symmetry and reciprocity claims need to be re-established from a real, correctly renormalized action.**

   The paragraph following equation (26) argues that P and T invariance follows because separate-sector $\mathcal{G}$ terms occur in squares or complex-conjugate pairs. That conclusion cannot survive unchanged once the missing invariants $F^{(1)} \cdot F^{(2)}$ and $F^{(1)} \cdot \widetilde{F}^{(2)}$ are restored. Treating g as a pseudoscalar spurion may make combinations such as $egF^{(1)} \cdot F^{(2)}$ formally invariant, but the paper must define the transformation of the fixed couplings, the dyon field and its antiparticle states, and both gauge potentials. A spurionic map between theories with g and $-g$ is not automatically a symmetry of one fixed theory.

   There are also signs of an incorrect differentiation of equation (25). Appendix equation (66) contains terms proportional to $i[X + \text{c.c.}]\widetilde{F}$, which are imaginary for real fields; differentiating $X + X^*$ with respect to a pseudoscalar invariant instead produces an $i(X - X^*)$ structure. Moreover, because the response coefficients are second derivatives of one real local Lagrangian, the mixed Hessian must obey reciprocity, such as $\rho^{(12)}_{ij} = \rho^{(21)}_{ji}$ and the analogous relation for σ. The coefficients in equation (37) are not generically symmetric under this exchange. These checks should be imposed when the field equations and constitutive tensors are rederived from the corrected polynomial action.

Minor comments and concrete corrections

1. The sector summation convention must be stated once and used consistently. Repeated (a) indices are summed in equation (6), appear inside nonlinear functions in equations (12)–(24), and are later declared not to be summed in equations (36) and (37). Ambiguity at precisely this point obscures the invalid eigenvalue step.

2. Equation (49) uses $\rho^{(a)}$ and $\sigma^{(a)}$, whereas only $\rho_0^{(a)}$ and $\sigma_0^{(a)}$ are defined. The same equation contains the malformed expression $B^2_{0(\bar{b})}$. Equation (40) alternates between the numeral zero and the letter "o" in background-field subscripts. These are not merely cosmetic because they occur in the central dispersion relation.

3. The renormalization discussion around equations (23) and (24) should display all quadratic counterterms, including the off-diagonal kinetic term, and state the subtraction conditions. The normalization of the two gauge fields and the definition of the renormalized charges otherwise remain underdetermined.

4. The symbols $E_{(a)}$ and $B_{(a)}$ are used both for nonnegative magnitudes and for signed or directed quantities inside sums. Vector expressions should be retained until a common direction and sign convention have been fixed.

5. The comparison of field scales after equation (54) should use the actual expansion parameter $|qB|/m^2$, rather than the pole of a denominator produced by the truncated constitutive relation. The latter is not an additional physical scale.

6. The manuscript would benefit from a compact transformation table for A_μ, B_μ, e, g, ψ, $F_{\mu\nu}$, and $G_{\mu\nu}$ under C, P, T, and the proposed $SO(2)$ duality. It should distinguish a symmetry at fixed couplings from a rotation of fields and charge parameters.

7. A careful copy-edit is needed. Examples include "pair productions," "an static monopole," the lower-case "lagrangian" after equation (26), and several missing spaces and inconsistent index placements. The physical and sector-specific meanings of "photon," "metaphoton," and "electromagnetic field" should also be kept distinct.

Required reconstruction

A viable replacement should begin from one consistent ghost-free action, diagonalize its gauge and matter charge structure, and compute the determinant from $\mathbb{F} = q_{(a)}F^{(a)}$. The weak-field action should be written in terms of the complete cross-invariant basis and accompanied by explicit kinetic-matrix renormalization conditions. The authors should then derive the quadratic response Hessian, verify reciprocity and the exact free charge-orthogonal mode, and analyze only orders controlled by the retained operators. Finally, any experimental birefringence proposal must specify the ordinary-matter source and detector couplings and use a background that is magnetic according to the paper's own physical-field definition. If additional matter or interactions are introduced so that two independent charge directions remain, the resulting model may support genuinely new mixed-sector optics; the present one-charge-vector theory does not.