

Referee Report

“A TQFT perspective on satellite constructions and toral decompositions”

arXiv:2607.11108

Recommendation: Reject in the present form, with encouragement to resubmit after a complete rederivation. The manuscript contains a potentially useful organization of satellite, splice, and JSJ constructions in three-dimensional TQFT, and the Hopf-chain transfer-matrix analysis appears to contain a publishable core. However, the principal Whitehead-double application is invalid because the Ising Whitehead operator used in Section 6.3 contradicts both the general operator formula (3.18) and the punctured S -matrix computed in equation (C.15). Correcting this single substitution removes the claimed even-odd effect, Figure 10, and equation (6.24). In addition, the manuscript confuses bordism maps with their dual right actions, does not yet define a presentation-independent two-dimensional defect theory, and contains several normalization and index errors in the entropy formulas. These are central correctness issues rather than matters suitable for a limited revision.

Summary and overall assessment

The paper studies link-complement states in the Reshetikhin–Turaev TQFT associated with a unitary, multiplicity-free modular tensor category. Section 2 fixes conventions for the torus Hilbert space and identifies the components of an n -boundary link-complement state with colored link invariants. Section 3 regards the complement of a pattern in a solid torus as a multilinear satellite operator. Its main prescription, equation (3.7), expresses that operator through the colored invariant of the pattern together with an auxiliary unknot. Connected-sum, torus-cabling, and Whitehead patterns are then treated as examples.

Section 4 extends this idea to splice pieces with several selected incoming and outgoing torus boundaries. Equation (4.10) is obtained by currying an ordinary link-complement tensor with the Hopf pairing. The author proposes that the JSJ decomposition should thereby be viewed as a tensor network of TQFT maps assigned to Seifert-fibered and hyperbolic pieces. Section 5 dimensionally reduces a Seifert-fibered piece along its circle fibers. Smooth surface topology is encoded by the Verlinde Frobenius algebra, exceptional fibers by point defects, and gluing maps between different Seifert fibrations by $SL(2, \mathbb{Z})$ loop defects. This produces the general Seifert operator (5.6) and the Hopf-chain state (5.15).

Section 6 applies the framework to entanglement. A single Seifert-fibered piece gives a GHZ-like state, and equations (6.1)–(6.5) discuss the many-boundary limit. For the open Hopf chain, the reduced state is controlled by the nonnegative transfer matrix

$$B_{ab} = \frac{|S_{ab}|^2}{S_{1a}S_{1b}}.$$

Its Perron–Frobenius vector determines the long-chain endpoint distribution. Finally, Section 6.3 applies the Whitehead operator at every boundary and claims a parity-dependent large-iteration entropy in the Ising category.

There is real merit in the organizing viewpoint. Equations (3.7) and (4.10) make standard TQFT functoriality concrete, the comparison between Seifert and hyperbolic JSJ pieces is physically

suggestive, and the transfer-matrix treatment of Hopf chains is more than a pictorial observation. Several basic checks of the latter work: equation (5.15) reduces to the Hopf state for two components and the Hopf-keychain state for three; B is indeed primitive in a unitary modular category; and the long-chain distribution has support on both invertible and non-invertible sectors. The main hyperbolic example, however, does not survive substitution of the paper’s own categorical data. Because that example supplies a substantial part of the claimed novelty and physical conclusion, the manuscript is not suitable for publication in its current form.

Correctness and logical structure

The satellite and Hopf-chain constructions are largely plausible after one interprets the displayed tensors as right-acting maps on complement bras. The present notation does not consistently make that interpretation: categorical source and target, geometric incoming and outgoing boundaries, and the dual action on bras are interchanged in several central equations. This can probably be repaired, but it must be repaired globally because the same convention controls boundary parametrizations, the order of modular transformations, and every later tensor contraction.

The Seifert reduction also provides a useful calculus, but equation (5.2) currently defines the reduced quantity by the original three-dimensional invariant and the manuscript explicitly leaves relations among equivalent Seifert presentations untreated. Thus Section 5 has not yet constructed the claimed defect TQFT as an independent, presentation-invariant theory. It is safer at present to regard it as a graphical calculus attached to chosen Seifert data and gluing maps.

The Hopf transfer matrix is the strongest sound calculation. A short proof omitted from the manuscript shows why its Perron–Frobenius limit is universal: the vacuum row and column of B are strictly positive, so

$$(B^2)_{ab} \geq B_{a1}B_{1b} > 0.$$

The endpoint argument is therefore convincing once the index and normalization errors below are corrected. By contrast, the Whitehead iteration uses the wrong matrix and its conclusions fail algebraically.

Novelty and significance

The paper should distinguish three levels of contribution more sharply. First, the existence of a linear map assigned to a pattern or splice complement follows directly from RT-TQFT functoriality and currying with the torus pairing; formulas (3.7) and (4.10) are useful explicit restatements, but not by themselves new topological theorems. Second, the interpretation of a graph-manifold plumbing as a two-dimensional Frobenius algebra with modular defects is a potentially valuable synthesis, provided presentation independence and framing conventions are supplied. Third, the analytic Perron–Frobenius treatment of Hopf-chain entanglement appears to be the most concrete new result, extending the numerical discussion cited as reference [9].

The manuscript already cites the prior fibered-link GHZ result [5], the large-party analysis [7], the numerical Hopf-chain work [9], and the Whitehead W -matrix literature [17]. A revised version should state precisely which propositions go beyond those references and should compare the proposed satellite operators with standard skein-theoretic satellite constructions and RT evaluations

of plumbed or graph manifolds. With the Whitehead parity effect removed, the author must reassess whether the surviving results meet the significance threshold of a broad high-energy theory journal. The Hopf-chain result may do so, but it should be promoted to a precise theorem with hypotheses, proof, and corrected finite-size formulas.

Major comments

1. The Ising Whitehead operator is wrong, invalidating the main result of Section 6.3.

Equation (3.18) defines

$$\mathcal{P}_{\mathcal{W}} = \sum_{a,c} |c\rangle\langle a| \frac{S_{aa}^c}{S_{1a}}.$$

Equation (C.15), in the ordered Ising basis $(\mathbf{1}, \sigma, \psi)$, states that $S_{ab}^{\sigma} = 0$ and

$$S_{ab}^{\psi} = \delta_{a\sigma} \delta_{b\sigma} e^{-i\pi/4}.$$

Consequently, direct substitution gives

$$\mathcal{P}_{\mathcal{W}} = |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{1}\rangle\langle\psi| + \sqrt{2}e^{-i\pi/4}|\psi\rangle\langle\sigma|,$$

not the term $\sqrt{2}e^{-i\pi/4}|\sigma\rangle\langle\sigma|$ printed in equations (6.22) and (C.16). Equation (C.15) is the structurally consistent side of this contradiction: the fusion rule $\sigma \otimes \sigma = \mathbf{1} \oplus \psi$ permits punctures $c = \mathbf{1}, \psi$, not $c = \sigma$, and $S_{\sigma\sigma}^{\mathbf{1}} = S_{\sigma\sigma} = 0$.

This changes the powers of the operator qualitatively. With $\alpha = \sqrt{2}e^{-i\pi/4}$,

$$\mathcal{P}_{\mathcal{W}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}, \quad \mathcal{P}_{\mathcal{W}}^2 = \begin{pmatrix} 1 & \alpha & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{P}_{\mathcal{W}}^l = \mathcal{P}_{\mathcal{W}}^2 \quad (l \geq 2).$$

Thus equation (6.23) is false: the normalized power does not tend to $|\sigma\rangle\langle\sigma|$. Moreover, the corrected operator makes the transformed Hopf-chain state a product already after one application. Indeed, only original labels $\mathbf{1}$ and ψ survive, and on that abelian subblock $Q_{ab} = 1$ and $S_{ab} = 1/2$. Equation (5.15) therefore gives, up to an overall constant,

$$Z(S^3 \setminus \nu(C^n))(\mathcal{P}_{\mathcal{W}})^{\otimes n} \propto (\langle\mathbf{1}| + \alpha\langle\sigma| + \langle\psi|)^{\otimes n}.$$

Its endpoint entropy is zero for every chain length, with no parity distinction. Figure 10, the even-odd discussion, and equation (6.24) must therefore be withdrawn or completely recomputed. The introductory and concluding claims about the distinct entanglement pattern generated by hyperbolic pieces must also be reassessed. This is not a typographical correction: it removes the paper's principal concrete hyperbolic example.

2. The manuscript must consistently distinguish a bordism map from its dual action on complement bras.

Equation (2.2) uses the standard convention for a bordism $M : \Sigma_1 \rightarrow \Sigma_2$. Equation (3.3) then displays

$$\bigsqcup_{i=1}^n T^2 \longrightarrow T^2,$$

while the accompanying prose calls the single outer torus incoming and the n pattern tori outgoing. With the standard source–target convention, those statements require the opposite arrow. Equation (3.4) calls the assigned map $\mathcal{V}_* \rightarrow \mathcal{V}_*^{\otimes n}$, whereas equations (3.6) and (3.7) display a tensor in $\mathcal{V} \otimes (\mathcal{V}^*)^{\otimes n}$, naturally a map $\mathcal{V}^{\otimes n} \rightarrow \mathcal{V}$. The former is the dual/right action of the latter, not literally the same functorial map.

The same reversal appears in Section 4. Equation (4.1) displays s tori mapping to r tori after defining r boundaries as incoming and s as outgoing; equation (4.2) instead has domain $(\mathcal{V}^*)^{\otimes r}$ and codomain $(\mathcal{V}^*)^{\otimes s}$; and equation (4.10) is algebraically a map $\mathcal{V}^{\otimes s} \rightarrow \mathcal{V}^{\otimes r}$. The formulas can be related by dualization, but that operation must be explicit.

The author should choose one categorical convention, mark orientation reversals on every boundary, and write separately $Z(W)$ and $Z(W)^*$. Equations (3.3)–(3.7), (4.1)–(4.10), and the product/comultiplication tensors in (5.4) should then be rechecked. Small identity-cylinder, Hopf-link, and connected-sum diagrams would make the convention unambiguous.

3. **Boundary parametrizations, slope conventions, and framing phases need a precise treatment.**

A satellite is not specified only by an unparametrized torus gluing. The meridian and preferred longitude of the pattern torus must be identified with those of the companion neighborhood, and the gluing of oriented three-manifolds is orientation reversing on the boundary. The paper uses an S -map to mediate this identification but does not state the exact matrix on homology, the role of S^{-1} versus S , or the possible charge-conjugation factor $S^2 = C$.

There is also a visible row–column mismatch. Equation (3.16) chooses U through the row-vector condition $(1, 0)U = (p/n, q/n)$, whereas equation (5.5) uses the column-vector condition $U(1, 0)^T = (\alpha, \beta)^T$. These conventions generally transpose the mapping-class action and affect equation (5.9), where line reversal is described as the linear dual. The author should give a single action on ordered meridian–longitude columns, specify the orientation-reversing gluing separately, and derive the corresponding action on bras.

Finally, the RT representation is projective unless the appropriate extended-surface or framing data are retained. Overall phases do not change a normalized entropy, but the paper writes the defect fusion equations (5.7)–(5.9) and operator identities as exact equalities. The framing or anomaly convention under which these equations hold should be stated, especially for iterated satellites where relative twists are part of the topology.

4. **Section 5 has not yet defined a presentation-independent two-dimensional defect TQFT.**

Equation (5.2) defines the reduced functor by the original three-dimensional invariant. The subsequent generators are useful, but the manuscript explicitly acknowledges after equation (5.5) that equivalent Seifert presentations, alternative fibrations of small Seifert manifolds, and the combination of regular marked fibers require additional relations that are not supplied. Nevertheless, the later discussion treats isotopy, defect fusion, and graph-manifold decompositions as if a well-defined defect bordism category and functor had been constructed.

This gap is important because the same graph manifold can have different Seifert data, different decompositions of a Seifert piece into pairs of pants, and nonunique fiber choices in exceptional cases. A genuine two-dimensional TQFT description must give generators and relations sufficient to prove that all such presentations yield the same three-dimensional operator, including the

projective modular relations and the dual of an orientation-reversed defect. The author should either supply this construction and a well-definedness proof, or recast Section 5 as a computational graphical calculus for a fixed presentation. The latter would still be useful, but the claims of a general effective defect TQFT would need to be narrowed.

5. The splice/JSJ statements need sharper hypotheses and a sphere-decomposition stage.

The JSJ theorem applies to irreducible manifolds. Link complements of split links are reducible, so one must first perform the sphere or prime decomposition and only then apply JSJ to the irreducible factors. The text moves directly from a general link complement to Seifert and hyperbolic JSJ pieces without treating this case.

The assertion following equation (4.10) that splice pieces can be “canonically re-embedded” as link complements in S^3 also needs a precise theorem and hypotheses. It is reasonable for the untwisted re-embeddings arising from the companionship structure of links in S^3 , but it is not a statement about an arbitrary compact manifold with torus boundary. For other ambient manifolds the manuscript itself introduces surgery data, at which point the resulting network depends on a surgery presentation rather than solely on the intrinsic JSJ decomposition. A revised version should distinguish: (i) classical link splicing in S^3 ; (ii) currying an arbitrary S^3 link state; (iii) intrinsic JSJ gluing with explicitly parametrized boundary tori; and (iv) an auxiliary surgery presentation. At present these are discussed too interchangeably.

The paper should also establish that the tori used in the Hopf-chain and Whitehead constructions are the minimal essential JSJ tori rather than merely tori in a chosen splice decomposition. That distinction matters for the claimed relation between entanglement and canonical JSJ structure.

6. The connected-sum formula is stated under conditions that are too weak.

Section 3.3.1 assumes that one component $\mathcal{L}_1$ has wrapping number one and each other component has wrapping number zero, and from this deduces the factorization (3.12). The displayed factorization requires a single meridional disk that meets $\mathcal{L}_1$ once and is disjoint from all other components, so that the auxiliary Hopf pair and the remainder are joined by the claimed connected sum. Individual minima in the definition of wrapping number do not by themselves exhibit such a simultaneous disk; link patterns can be collectively essential even when individual components admit simpler positions.

The author should either strengthen the hypothesis to the actual connected-sum pattern geometry, with a common disk or separating sphere, or prove that the componentwise conditions imply it. Without that step, equations (3.11)–(3.12) are not established at the advertised level of generality.

7. The Seifert large-boundary limit is misnormalized and lacks a necessary support condition.

Equation (6.2) does not define normalized probabilities as printed. If

$$A_a = \prod_{i=1}^k (SU_{(\alpha_i, \beta_i)})_{a1},$$

then its abelian denominator is written as $(\sum_b \text{abelian} |A_b|)^2$, whereas normalization requires $\sum_b \text{abelian} |A_b|^2$.

More conceptually, equations (6.2) and (6.5) assume that at least one invertible sector has nonzero coefficient. Defect factors or the inner JSJ network can make every abelian coefficient vanish. In that case the displayed denominator is zero and the large- n limit localizes instead on the smallest quantum dimension among the sectors that remain in the support. The correct statement is support-dependent: if $d_{\min}$ is the minimum d_a for which the total coefficient is nonzero, only sectors with $d_a = d_{\min}$ survive, with weights given by their squared coefficients. Abelian dominance is the special case $d_{\min} = 1$.

The same qualification is required for the claim after equation (6.3) about an “outermost” Seifert piece. The author should define what outermost means for a general JSJ graph, identify which external boundaries are included in the bipartition, and state the nonvanishing assumptions on the coefficients λ_a .

8. The Hopf-chain result is promising, but several displayed formulas must be corrected.

Equation (6.11) defines p_a while summing over a and leaving b free. A consistent version is

$$p_b = \frac{\sum_a (B^{n-3})_{ab} / (S_{1a} S_{1b})}{\sum_{c,d} (B^{n-3})_{cd} / (S_{1c} S_{1d})}.$$

Immediately after equation (6.12), the expression $B^n \simeq \lambda^n \xi^T \xi$ is a scalar rather than a matrix and omits normalization. It should read

$$B^n \sim \lambda^n \frac{\xi \xi^T}{\xi^T \xi}$$

unless ξ has already been normalized. The claim of primitivity should be accompanied by the short $B^2 > 0$ proof given above.

The pointed-category paragraph also uses the wrong normalization: with N simple objects, $|S_{ab}| = 1/\sqrt{N}$, not one. The conclusion $B_{ab} = 1$ and the uniform distribution remain correct. The manuscript should avoid reusing N here after using $\mathcal{N}$ for the number of non-abelian objects.

Appendix B has further normalization errors. In equation (B.1), the trace denominator retains a primed multi-index; it should be

$$\sum_{\vec{a}} |A_m(\vec{a})|^2 H_{n-m}(a_m, a_m).$$

The denominator of equation (B.7) drops the factor $|S_{a_m b}|^2$ required by tracing its numerator, while equation (B.10) restores the corresponding factor. In equation (B.13), $g_b = \sum_a (F^{m-1})_{ab}$ scales as λ^{m-1} , not λ^m ; this common power cancels after normalization but should still be correct. The author should rederive equations (B.1)–(B.14) with every summation index displayed and verify positivity and unit trace at each step.

Finally, the validity ranges should be stated. Equations (6.8)–(6.11) contain B^{n-3} and require $n \geq 3$; the Hopf-link case $n = 2$ must be treated separately. Equation (B.4) contains B^{k-2} and requires $k = n - m \geq 2$, whereas the text advertises a general contiguous bipartition that includes $k = 1$.

9. The Whitehead rank statement should be separated from the invalid Ising iteration and phrased precisely.

The rank bound (6.17) remains plausible and useful. Equation (3.22) shows that all invertible input columns of $\mathcal{P}_W$ are proportional to the vacuum output, so the image has dimension at most one plus the number of non-invertible simple objects. This is a collapse of the abelian input subspace to one direction, not an annihilation of every non-vacuum abelian contribution. Coherent amplitudes from those inputs can remain in the vacuum column.

A revised paper should prove the rank-to-entropy bound carefully for a nonunitary local map, specify the framing of the iterated Whitehead pattern, and then choose a categorical example whose correctly evaluated operator has genuinely nontrivial powers. The recursion (6.19)–(6.21) should be derived in components and accompanied by reproducible data. Its present use with the wrong Ising matrix cannot support any conclusion about generic hyperbolic pieces.

Minor comments

1. Figure 4 attributes wrapping number one to the companion. Wrapping number is a property of the pattern in the solid torus.
2. Equation (3.8) inserts $\nu(\mathcal{O} \cup \mathcal{L})$ inside a colored link invariant. The tubular-neighborhood symbol should not be present there.
3. The prose after equation (5.4) reverses the Frobenius operations displayed algebraically. The tensor $|c\rangle\langle ab|$ is multiplication, $|\widetilde{a}\widetilde{a}\rangle\langle\widetilde{a}|$ is comultiplication, $|\mathbf{1}\rangle$ is the unit, and $\langle\mathbf{1}|$ is the counit.
4. The text before equation (5.15) refers to the “Hopf keychain complement” although the equation gives the general Hopf-chain complement.
5. The claim that a torus-link complement has two exceptional fibers should allow the degenerate cases in which one multiplicity is one and the corresponding fiber is regular.
6. The open Hopf chain $\mathcal{C}^n$ should be defined by an explicit oriented link diagram, including crossing signs and framings. The phrase “chain link” is used for several inequivalent families in the topology literature.
7. Equation (3.21) should state explicitly how orientation reversal and conjugation of the label b enter the Whitehead symmetry, since that identity is later used to compare left and right kernels.
8. The identity immediately before equation (A.3) contains $\theta_a\theta b$ rather than $\theta_a\theta_b$.
9. Appendix A should specify the gauge and normalization of the F -symbols used in (A.1)–(A.5). Although the punctured S -matrix is gauge-invariant, intermediate entries are not.
10. The manuscript contains several correctable typographical problems, including “is is,” “an one-sided loop,” “consequence,” “suppresion,” and subject–verb disagreement after equation (6.2). The bibliography also contains escaped text such as “Algebraic amp; Geometric Topology.”
11. The statement that the left and right nullspaces of the Whitehead operator are isomorphic should say whether a canonical isomorphism is intended. Equal dimensions hold for every square matrix

and do not by themselves follow from the Whitehead symmetry.

12. The paper would benefit from stating its main claims as propositions with explicit hypotheses. At present no theorem, proposition, or lemma is isolated, which makes it difficult to separate exact RT identities, claims valid only up to local unitaries or framing phases, and asymptotic statements requiring nonzero support.

Recommendation

The paper has a worthwhile conceptual program and a promising Hopf-chain calculation, but correctness, novelty, and significance must be judged on the submitted results. The incorrect Ising operator eliminates the main Whitehead-double conclusion, while the central variance conventions and several entropy normalizations remain unresolved. I therefore recommend rejection in the present form. A substantially reconstructed submission could merit fresh consideration if it (i) replaces Section 6.3 and Figure 10 with a correct calculation, (ii) fixes the bordism/dual conventions and gluing data throughout, (iii) makes the two-dimensional defect formalism presentation-independent or narrows its claims, (iv) repairs and verifies all entropy formulas, and (v) positions the surviving Hopf-chain result precisely relative to the existing literature.