Referee Report

Evading Cosmological Strong Coupling in Non-minimally Coupled Vector Gravity

arXiv:2607.11125

Recommendation: Major revision. The manuscript presents a potentially useful observation: after adding the operator $(\nabla_\mu X^\mu)^2$ to the non-minimally coupled vector model, the authors find a nonzero-condensate de Sitter point at which the displayed high-momentum kinetic minors and scalar propagation speeds are positive. The background calculation and several benchmark numbers are internally consistent. However, the present paper does not establish that the quadratic system has positive energy, does not provide the matrices needed to verify its central scalar calculation, and does not determine the strong-coupling scale advertised in the title. Moreover, the discussion of the exactly vanishing-condensate branch misses a defective principal symbol and an unbounded scalar Hamiltonian, and equation (29) contains a definite factor-of-three error. The core idea may be publishable if a full Hamiltonian, hyperbolicity, and finite-momentum analysis confirms it, but the current evidence is insufficient for publication in PRD or JHEP.

Summary and overall assessment

The paper studies the four-parameter vector–tensor action in equation (1), consisting of the Maxwell term, a Proca mass, the two non-minimal couplings $RX_\mu X^\mu$ and $R_{\mu\nu}X^\mu X^\nu$, and the additional divergence-squared interaction proportional to ξ_3. Equations (3) and (4) give the homogeneous equations on a flat FLRW background with a temporal vector condensate. The nonzero-condensate de Sitter branch obeys equation (6).

For scalar perturbations, the lapse and shift are integrated out and the two remaining variables are assembled into the reduced quadratic action (10). The authors state high-momentum expressions for K_{11} and $\det K$ in equations (13) and (14), derive the frozen-coefficient quartic dispersion relation (18), and impose the frequency criteria (22). The benchmark (23) is then reported to have two positive scalar speeds. Tensor and vector ultraviolet conditions are summarized in equations (25) and (26). Finally, the authors derive a homogeneous fixed-point equation, construct an autonomous background equation (32), and show one trajectory approaching the de Sitter point while the displayed ultraviolet diagnostics remain positive.

There is a worthwhile result inside this analysis. Using the formulas printed in the manuscript, the point (23) gives

$$\Xi = 11, \qquad \frac{K_{11}}{a^3} = 4, \qquad \frac{\det K}{a^6} = 0.36,$$

as well as $G_T = 1.2$ and $c_T^2 = c_V^2 = 1$. Equations (27), (28), and (31) consistently give $\xi_4 m_X^2/M_{\rm Pl}^2 = -0.12$ and $\Lambda/M_{\rm Pl}^2 = 0.033$. Equations (32)–(38) are also consistent with the background equations and with the fixed point. Thus the proposed nonzero-condensate example is not merely the result of an obvious substitution error.

The difficulty is that these checks do not reach the main physical conclusion. The scalar matrices K_{ij}, L_{12}, and M_{ij} are absent, so the quoted speeds and finite-momentum spectrum cannot be checked. More importantly, positive kinetic minors and real positive frequencies are not sufficient for a positive-energy gyroscopic system. The exact $A_0 = 0$ limit furnishes a direct counterexample to the health criterion used in the paper. Finally, lifting a zero sound speed at quadratic order is not by itself a calculation of the interaction cutoff. These are central issues of correctness and interpretation, rather than requests for additional phenomenology.

Major comments

1. **The health criteria in equations (12) and (22) do not establish a bounded Hamiltonian.**

   For frozen coefficients, the antisymmetric L_{12} term in equation (10) is gyroscopic. In terms of the velocities, the corresponding quadratic Hamiltonian is

   $$\mathcal{H}^{(2)} = \dot{\boldsymbol{\psi}}^T K \dot{\boldsymbol{\psi}} + \boldsymbol{\psi}^T M \boldsymbol{\psi}.$$

   Consequently, $K > 0$ and two real positive values of ω^2 do not by themselves imply positive energy. A negative-definite M can be stabilized by a sufficiently large gyroscopic term and still satisfy a real-frequency test, while carrying negative-energy (Krein-signature) modes. Equation (22) imposes $\det M > 0$ through $\mathcal{C}/\mathcal{A} > 0$, but it does not distinguish positive-definite from negative-definite M. The paper should calculate the canonical Hamiltonian, the symplectic norm or residue of each normal mode, and the relevant gradient-energy matrix at the purported healthy point. At minimum, the authors should show $M_{11} > 0$ together with $\det M > 0$ in the ultraviolet, or give an equivalent positive-mode-energy test.

   The equality case in the last condition of equation (22) also requires special care. A repeated real characteristic is not sufficient for a well-posed Cauchy problem; the principal symbol must possess a complete, uniformly conditioned eigenbasis. This omission is not academic, because it is precisely what fails on the $A_0 = 0$ branch discussed below.

2. **The exactly vanishing-condensate branch is not a regular healthy limit.**

   At $A_0 = 0$, the vector scalar perturbations decouple from the metric in the principal high-frequency part. Let $u = \delta A_0$, $v = (k/a)\delta A$, and $p = k/a$. Up to an immaterial Fourier phase convention, equation (1) gives

   $$\mathcal{L}_{\text{prin}} = \frac{1}{2}\left|\dot{v} + pu\right|^2 - \frac{\xi_3}{2}\left|\dot{u} - pv\right|^2.$$

   This reproduces the manuscript's $K_{11} = -\xi_3 a^3/2$ and $K_{22} = a^3/2$. For $\xi_3 < 0$ the velocity Hessian is positive, but the static gradient contribution to the Hamiltonian is negative and unbounded. The characteristic determinant is proportional to

   $$-\xi_3(\omega^2 - p^2)^2.$$

   For $\xi_3 \neq 1$, the characteristic matrix has rank one at each double light-cone root, hence only one eigenvector for two coincident scalar characteristics. This is a Jordan or dipole mode, with secular high-frequency solutions, rather than a strongly hyperbolic two-mode system. At $\xi_3 = 1$ the principal equations diagonalize, but the temporal kinetic coefficient has the ghost sign.

   The mass terms do not cure this problem. On a de Sitter $A_0 = 0$ background the local effective vector mass is

   $$m_{\text{eff}}^2 = \xi_4 m_X^2 + 3H^2(4\xi_1 + \xi_2).$$

   The two local scalar poles behave as $\omega^2 = p^2 + m_{\text{eff}}^2$ and $\omega^2 = p^2 + m_{\text{eff}}^2/\xi_3$. For the no-ghost Hessian choice $\xi_3 < 0$, one is tachyonic for either sign of nonzero m_{eff}^2; when $m_{\text{eff}}^2 = 0$, the defective double root remains. Thus the paragraph following equation (17) establishes only positivity of the velocity Hessian, not that the branch is *regular*. The paper must replace that conclusion with a Hamiltonian and strong-hyperbolicity analysis. The same mode-energy test should then be applied to the nonzero-condensate point, where distinct quoted speeds make strong hyperbolicity plausible but do not prove positive energy.

3. **The central scalar perturbation calculation is not reproducible from the manuscript or its Supplemental Material.**

   Equation (10) is the foundation of every principal result, yet none of the entries of K_{ij}, M_{ij}, or L_{12} is supplied. Equation (11) is only schematic, and the Supplemental Material replaces the lapse and shift constraints by unspecified coefficients h_i and m_i. It therefore does not permit a reader to reproduce equations (13)–(15), the two speeds in equation (24), the stability diagnostics in Figure 1, or the claimed open region. This is especially problematic because the claimed cure is a change in the constraint and kinetic structure.

   The authors should provide either the complete matrices and constraint denominator or a machine-readable ancillary derivation. The benchmark should be accompanied by analytic or high-precision expressions for the two ultraviolet speeds and by the kinetic eigenvalues or residues. The exact matrices should also be evaluated over momentum, rather than only in the $p \to \infty$ limit.

   There are two additional inconsistencies in the purported derivation. First, equation (10) defines $\psi_2 = (k/a)\delta A$, whereas Supplemental equations (46)–(50) use δA with the same coefficients. Since k/a is time dependent, the two systems differ by terms proportional to H unless the matrices are correspondingly redefined. Second, exact variation of equation (10) with time-dependent L_{12} gives, for the first component,

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(K_{11}\dot{\psi}_1 + K_{12}\dot{\psi}_2\right) + L_{12}\dot{\psi}_2 + \frac{\dot{L}_{12}}{2}\psi_2 + M_{11}\psi_1 + M_{12}\psi_2 = 0,$$

   with the analogous opposite-sign terms in the second equation. Supplemental equations (46) and (47) put all of $L_{12}\psi_j$ inside the time derivative, doubling the $\dot{L}_{12}$ contribution. This does not change the leading frozen-coefficient WKB polynomial, but the displayed equations are not the exact Euler–Lagrange equations claimed in the text.

4. **Equation (29) contains a definite factor-of-three error.**

   As printed, the restoring coefficient in equation (29) is

   $$3\beta_A^2 \frac{3(4\xi_1 + \xi_2)(2\xi_1 - \xi_2 - 3\xi_3)}{\xi_3 - \Xi\beta_A^2}.$$

   At the point (23), where $\Xi = 11$ and $\beta_A^2 = 0.2$, this equals

   $$3(0.2)\frac{3(4)(-1)}{1 - 11(0.2)} = 6,$$

   not 2. The printed equation would therefore give the exponents $(-3 \pm i\sqrt{15})/2$, not the two decays in equation (30).

   Direct linearization of equations (3) and (4) gives instead

   $$\Delta A_0'' + 3\Delta A_0' + \frac{3\beta_A^2(4\xi_1 + \xi_2)(2\xi_1 - \xi_2 - 3\xi_3)}{\xi_3 - \Xi\beta_A^2}\Delta A_0 = 0.$$

This coefficient is 2 at the benchmark and is consistent with equation (30) and with the linearization of the autonomous equation (32). Thus equation (29) appears to contain one extra factor of 3. Although the corrected fixed point remains attractive, this error must be fixed and all formulas derived from the homogeneous linearization should be rechecked.

5. **The manuscript does not calculate the strong-coupling scale or the finite momentum at which the small-condensate ghost appears.**

   Equations (13)–(17) establish only the sign of the $p \to \infty$ kinetic determinant. The assertion that the ghost scale moves to arbitrarily high momentum as $A_0 \to 0$ is inferred from the schematic dependence in equation (11); no expression for $p_{\text{ghost}}(A_0)$ is given. The exact constraint denominator could contain poles or kinetic zeroes before the asymptotic regime is reached. Moreover, the interaction cutoff after canonical normalization can itself depend singularly on A_0. A comparison between p_{ghost} and a background-independent cutoff is therefore not justified.

   The nonzero-condensate branch presents a related issue. Equation (14) contains an explicit H^2 factor, so the normalization of the smaller scalar kinetic direction can become parametrically small when the same dimensionless condensate is placed at a realistic cosmological Hubble scale. The benchmark $H/M_{\text{Pl}} = 0.1$ does not probe this hierarchy. The full matrices are needed to identify the canonically normalized eigenmode and its interactions.

   To support the title, the authors should derive the leading cubic interactions, canonically normalize both scalar modes, and estimate the background-dependent strong-coupling scale. They should also give the full finite-p kinetic eigenvalues and locate $p_{\text{ghost}}(A_0)$. If this is beyond the intended scope, the title, abstract, and conclusions should be narrowed to the precise result actually shown: the $c_s^2 = 0$ quadratic degeneracy of the earlier model is lifted at one nonzero-condensate de Sitter point.

6. **Ultraviolet and isotropic homogeneous tests do not establish complete cosmological stability.**

   The rational momentum dependence described in equation (11) can allow finite-momentum kinetic sign changes, constraint poles, or tachyonic bands even when both the $p = 0$ and $p \to \infty$ limits look acceptable. The paper should show the full scalar kinetic eigenvalues and dispersion relations for $0 \leq p < \Lambda_{\text{EFT}}$ along the relevant background trajectory. The vector sector is treated only through its kinetic coefficient and ultraviolet sound speed in equation (26); its mass and possible low-momentum instability are not given. The same applies to the non-derivative scalar stability matrix.

   The heading *Homogeneous stability* is also too broad. Equations (29)–(32) perturb only the isotropic variables A_0 and H. The direct predecessor, reference [12], found additional directional and unstable modes on Bianchi I backgrounds. A temporal vector cosmology should therefore be tested against homogeneous spatial-vector and shear perturbations, or the claim should be explicitly restricted to attraction within the isotropic FLRW subspace.

   Finally, Figure 1 does not demonstrate a numerical basin. It shows one time series for A_0 and H, plus the diagnostics along that same trajectory. The initial values, in particular A_0', are not stated, and no phase-space boundary or family of initial conditions is shown. Linear attraction plus continuity supports a local neighborhood, but the claimed finite basin in which all stability conditions remain positive should be documented with a phase portrait or a scan over initial data, together with the integration method and tolerances.

7. **The physical status and novelty of the additional scalar require a clearer comparison with vector–tensor theory.**

   The operator $(\nabla_\mu X^\mu)^2$ changes the Hessian and intentionally propagates a fourth vector polarization; the model is not generalized Proca, whose degeneracy removes that mode. The paper should give a nonlinear constraint and degree-of-freedom count and explain why this mode, often regarded as the unwanted vector scalar, has a bounded physical Hilbert space on the proposed background. This is particularly important in light of the Hamiltonian problem on the $A_0 = 0$ branch.

   The literature discussion should go beyond the single generalized-Proca citation. General quadratic vector–tensor actions and their stability have been studied, for example in J. Beltran Jimenez and A. L. Maroto, JCAP 02 (2009) 025, arXiv:0811.0784, while the degeneracy classification directly relevant to the fourth mode is given in R. Kimura, A. Naruko, and D. Yoshida, JCAP 01 (2017) 002, arXiv:1608.07066. The authors should locate equation (1) within these classifications and state precisely what is new relative to known divergence-squared vector cosmologies and extended vector–tensor theories.

   There is also no symmetry or power counting that explains the truncation in equation (1). Once the vector $U(1)$ is broken, operators such as $(X_\mu X^\mu)^2$ are allowed and can affect a background with $\bar{A}_0/M_{\rm Pl} \simeq 0.45$. Since the interpretation rests on an EFT cutoff, the assumed organizing principle and radiative stability of the four-parameter action should be stated.

8. **The significance claims should be calibrated to the demonstrated solution.**

   The explicit point is a useful existence test, but it is not yet a realistic dark-energy solution. Equation (28) shows that the acceleration is supported primarily by an explicit cosmological constant: $\Lambda/M_{\rm Pl}^2 = 0.033$ compared with $3\bar{H}^2/M_{\rm Pl}^2 = 0.030$. No matter era, transition to the fixed point, or self-accelerating limit is exhibited. The benchmark also has $c_{s+}^2 \simeq 1.14$. The manuscript correctly notes that superluminality is not the same as a linear instability, but the only explicit healthy point therefore raises an additional obstruction to a standard Lorentz-invariant ultraviolet completion.

   The authors should state whether subluminal healthy points exist and whether the same mechanism survives at a phenomenologically relevant hierarchy of H, m_X, and the cutoff. If the aim is solely a proof of principle, the conclusions should say so and should avoid presenting the result as a dynamical explanation of late-time acceleration.

Minor comments

1. In equation (11), the coefficients $c_1, \ldots, c_4$ must carry matrix-element labels; a single ratio cannot describe all entries of K_{ij} with the same four functions.

2. The surface $\xi_3 - \Xi\mathfrak{A}_0^2 = 0$ is not merely a *boundary of this dynamical chart.* Equation (14) shows that the ultraviolet kinetic determinant vanishes there, while equation (32) becomes singular. It is a kinetic-degeneracy or strong-coupling surface. At the benchmark it lies at $\mathfrak{A}_0^2 = 1/11$ and separates the healthy fixed point at 0.2 from the near-zero regime. This physical interpretation should be stated.

3. Below equation (26), the phrase *coincide with the speed of sound* should be replaced by *are*

luminal or *coincide with the speed of light*. In the same paragraph, it would be useful to state explicitly that the benchmark gives $G_T = 1.2$.

4. In the discussion below equation (32), $a_0^2 = 1/11$ should read $\mathfrak{A}_0^2 = 1/11$. The plot labels M_P and $\mathcal{N}$ should be made consistent with the notation $M_{\rm Pl}$ and $\mathcal{N}$ in the text.

5. References [11] and [17] have different titles and author lists but are assigned the same DOI and arXiv identifier, 10.1103/PhysRevD.94.044024 and arXiv:1605.05066. One entry is incorrect. The screening reference should also be checked against the statement in the Introduction.

6. Since the dynamics depends on the product $\xi_4 m_X^2$, the count of independent parameters should distinguish the dimensionless couplings from the mass and cosmological scales.

7. The phrase *branch (6)* should be accompanied by the Friedmann condition (28), since equation (6) alone is only one algebraic relation. The allowed signs of ξ_4 and m_X^2 should also be specified.

8. The manuscript should distinguish absence of an ultraviolet gradient instability, positive mode energy, absence of tachyonic growth, and weak coupling. These are separate conditions and are currently used too interchangeably.

Recommendation

The paper contains a plausible and potentially interesting quadratic high-frequency example on a nonzero temporal-vector condensate. The corrected homogeneous equation retains an attractive fixed point, and the displayed kinetic minors at that point are positive. However, publication requires substantially more than these checks. The authors must establish positive mode energy and strong hyperbolicity, provide the omitted scalar matrices, analyze the full momentum range and anisotropic homogeneous modes, and either calculate the interaction cutoff or sharply narrow the strong-coupling claim. I therefore recommend **major revision**. If the Hamiltonian or residue analysis reveals that the nonzero-condensate modes are only gyroscopically stabilized negative-energy modes, the main conclusion would fail and the paper would require a fundamental reformulation rather than revision.