Referee Report

"Conjugate Boundary Conditions, Kaluza–Klein Fermions, and an Extended Seesaw Model"

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Recommendation: Major revision. The manuscript contains a useful explicit treatment of fermions obeying a conjugate boundary condition (CBC). In particular, the free Kaluza–Klein calculation supports the stated spectrum: one Majorana zero mode and, for vanishing Scherk–Schwarz phase, two degenerate Majorana fields at every nonzero level that can be assembled into one Dirac field. The CBC-compatible mixing with a chiral-orbifold fermion is also a potentially useful model-building observation. However, the central interpretation of lepton-number violation is not established by a symmetry or spurion analysis and is in tension with the paper's own low-energy mass matrix, whose light Majorana mass is proportional to M_Φ and vanishes when $M_\Phi = 0$. The displayed mass texture also omits other neutral-fermion Majorana operators allowed by the stated boundary conditions without providing a protecting symmetry. The proposed model is analyzed only in its zero-mode truncation, the complex-flavor diagonalization is not correct as written, and the phenomenological discussion does not include the compactification scale, KK tower, flavor structure, or a derivation of the claimed electron EDM. These issues affect the principal claims of novelty and viability and must be resolved before publication can be considered.

Summary and overall assessment

Section 2 reviews the CBC on $\mathcal{M}^4 \times S^1/Z_2$. The reflection identifies a five-dimensional spinor with its four-dimensional charge conjugate. Equations (6)–(9) explain why a naive five-dimensional charge conjugation must be accompanied by γ_5 in order to compensate the sign of the y derivative. Equations (12)–(18) then compare the CBC with the ordinary chiral orbifold projection and derive a boundary-compatible linear combination

$$\bar{\eta}(\psi - PP'\gamma_5\psi^{c}).$$

The section also sketches a charged-scalar version in equations (19)–(20).

Section 3 gives the most complete part of the paper. For $P = +1$, the real Majorana combination ψ_{M} is even and the imaginary combination $\psi_{\bar{\mathrm{M}}}$ is odd. Their cosine and sine wavefunctions lead to

$$m_n^2 = m^2 + \frac{n^2}{R^2}.$$

The first-order y derivative mixes the two Majorana fields at nonzero n. Equations (38)–(40) remove this mixing with an axial rotation in the two-field space, after which equations (41)–(43) identify each degenerate pair with a Dirac spinor. The zero mode remains a single Majorana field with mass $|m|$. The comparison with the general Scherk–Schwarz result of reference [11] correctly notes that the degeneracy is special to zero phase.

Section 4 applies the construction to a bulk lepton sector. The Standard Model lepton fields and a singlet N obey chiral orbifold projections, while a second singlet Φ obeys the CBC. Equation (50) includes the CBC-compatible N–Φ mixing and an ordinary bulk mass M_Φ for Φ. Retaining zero modes gives the familiar symmetric matrix

$$M = \begin{pmatrix} 0 & M_\nu^{\mathrm{T}} & 0 \\ M_\nu & 0 & M_N \\ 0 & M_N^{\mathrm{T}} & M_\Phi \end{pmatrix},$$

and equation (59) gives

$$m_\nu = M_\nu^{\mathrm{T}} (M_N^{\mathrm{T}})^{-1} M_\Phi M_N^{-1} M_\nu.$$

For $M_N \sim M_\Phi \sim 10\,\mathrm{TeV}$ and $M_\nu \sim 1\,\mathrm{MeV}$ this is of order $0.1\,\mathrm{eV}$. The paper attributes lepton-number violation to M_N, rather than to M_Φ, assigns Φ zero lepton number in Table 1, and gives order-of-magnitude estimates of non-unitarity and an electron EDM.

The mode calculation is clear enough to be useful, and its central free-spectrum result appears algebraically sound. The distinction between a global five-dimensional fermion number and an accidental quadratic symmetry of a fixed KK level is also worth spelling out. By contrast, the physical interpretation and model application are presently incomplete. A charge label in Table 1 does not by itself identify the source of observable $\Delta L = 2$ processes, and the zero-mode matrix is exactly the standard inverse-seesaw matrix that the manuscript cites. Moreover, most conclusions about viability are drawn before the parts specific to an extra-dimensional model—the KK tower, R^{-1}, boundary gauge sector, cutoff, and localization backgrounds—have been included.

Correctness and logical structure

The derivation through equation (43) has a coherent logical structure. The cosine/sine spectrum obtained from the second-order equation agrees with the subsequent diagonalization of the first-order quadratic form. With

$$\overline{\Gamma}_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \qquad m\mathbf{1} + i\frac{n}{R}\overline{\Gamma}_5 = m_n e^{2i\theta_n \overline{\Gamma}_5},$$

and $\tan 2\theta_n = (n/R)/m$, the axial field redefinition preserves the kinetic term and yields the mass m_n. It also preserves the Majorana reality condition: $i\gamma_5$ times a Majorana field is again Majorana. Thus the paper's free-theory conclusion is more than a heuristic counting statement.

Several qualifications are nevertheless needed. The initial use of a four-dimensional Dirac equation for $\psi_{\mathrm{M}}^{(n)}$ before the coupled first-order system has been diagonalized should be phrased only as a mass-squared separation. Equation (35) is false for the advertised $n = m = 0$ sine mode, since $S_0 = 0$ and is not normalized. The accidental $U(1)$ of a nonzero level is a symmetry of the diagonal quadratic action, not a local five-dimensional symmetry; interactions can mix levels as well as split the pair. These are repairable points.

The more substantial logical gap begins with the lepton-number discussion. The paper infers violation from the inability to assign a vector charge to a four-component CBC Majorana field. That is not sufficient: even in the ordinary type-I seesaw, the Majorana completion has no definite charge, while one still diagnoses lepton number by assigning a charge to the independent Weyl field and treating the Majorana mass as a spurion. The same two-component analysis must be done here, including the boundary condition and the entire KK tower. In the zero-mode truncation, setting $M_\Phi = 0$ restores the usual inverse-seesaw $U(1)$ and makes the light eigenvalue exactly zero. Consequently, the assertion that M_N, but not M_Φ, is the physical source of lepton-number violation does not follow from equations (56)–(60).

Novelty and significance

The paper should distinguish its three possible contributions. First, the general CBC spectrum, including equation (44), was already derived in reference [11]. At zero Scherk–Schwarz phase, pairing the n and $-n$ Majorana eigenstates into a Dirac field is a direct consequence of their degeneracy. The present explicit axial-rotation derivation is useful, but the manuscript must state precisely what result, rather than what presentation, is new. The attribution should also be corrected: the CBC in equation (5) is the $P1$–$R2$ case in the notation of reference [11], not $P1$–$R1$ as stated in the footnote.

Second, the boundary-compatible interaction in equation (18) may be the most original structural observation. Its relation to the earlier bulk-seesaw and extra-dimensional constructions in references [14] and [16] needs a more detailed comparison. In particular, the manuscript should identify whether those works contain the same zero-mode operator after field redefinitions and whether the proposed charged-scalar completion imposes genuinely new selection rules.

Third, the low-energy mass matrix (56) is, as the authors acknowledge, exactly the inverse-seesaw matrix. Relabeling which entry is said to violate lepton number is not by itself a new physical model. A publishable distinction would require a basis-independent consequence: a selection rule for a $\Delta L = 2$ amplitude, a different radiative stability property, a constrained flavor texture, or a characteristic KK spectrum/coupling relation. None is presently demonstrated. With $M_\Phi \sim M_N$, equation (59) reduces parametrically to the ordinary scaling $m_\nu \sim M_\nu^2/M_N$; the construction obtains 0.1 eV only after assuming an MeV Dirac mass. Thus it does not currently inherit the usual inverse-seesaw advantage of small neutrino mass with appreciable active–sterile mixing. The significance assessment should be revised accordingly.

Major comments

1. **The claimed origin of lepton-number violation is not established and is not invariant under the low-energy charge assignment.**

   The statements surrounding equations (45)–(49), Table 1, equations (56)–(60), and the Summary identify M_N as the source of lepton-number violation and declare the Majorana entry M_Φ lepton-number conserving. In a left-handed two-component basis (ν_L, N_R^c, Φ_L), however, the same mass matrix admits the standard assignment

   $$L(\nu_L) = 1, \qquad L(N_R^c) = -1, \qquad L(\Phi_L) = 1.$$

   Both M_ν and M_N then conserve this $U(1)$, while M_Φ violates it by two units. More decisively, for one generation the characteristic matrix with $M_\Phi = 0$ has an exactly massless eigenstate and a Dirac pair with masses $\pm\sqrt{|M_N|^2 + |M_\nu|^2}$. Equation (59) is linear in M_Φ and the low-energy Weinberg operator vanishes when M_Φ does. These are the standard signatures that M_Φ is the $\Delta L = 2$ spurion of the zero-mode theory.

   The five-dimensional CBC may obstruct lifting this accidental four-dimensional charge to a vector symmetry of the complete tower. If that is the intended distinction, it must be shown rather than encoded by assigning $L(\Phi) = 0$ in a table. The authors should formulate the transformations of the independent Weyl components, the boundary conditions, all bulk terms, and the KK modes under a candidate $U(1)$; identify the enhanced symmetry limits; and compute at least one basis-independent $\Delta L = 2$ quantity. In particular, they should determine whether KK exchange generates lepton-number violation when $M_\Phi = 0$. The abstract and conclusions must then describe the result of that analysis. As written, the claimed contrast with the conventional inverse seesaw is a convention, not a demonstrated physical distinction.

The same issue invalidates the motivation in equations (45)–(47). A zero-mode Yukawa coupling to the right-handed component of a Majorana field is the ordinary type-I seesaw coupling; it can conserve an accidental lepton number when the Majorana mass is removed. Calling the Yukawa itself "tree-level LNV" does not establish a phenomenological problem. The manuscript should exhibit the allegedly dangerous amplitude or revise this argument.

2. **The model does not yet provide a complete extra-dimensional calculation.**

   Equation (50) is presented as a UED model, but all quantitative conclusions after equation (53) retain only zero modes. No value or allowed range is given for R^{-1}, no cutoff is specified for the nonrenormalizable five-dimensional Yukawa theory, and the gauge-field boundary conditions and KK parity are absent. The heavy neutral scale $M_N \sim M_\Phi \sim 10\,\mathrm{TeV}$ cannot be assessed independently of R^{-1} and the cutoff. Nor is it enough to state that nonzero modes are subdominant when the very CBC interaction under study couples the chiral and Majorana towers.

   A revised paper should derive the nonzero-level neutral-fermion mass matrices and interaction vertices, identify which KK number or KK parity selection rules survive, and state a controlled regime in which integrating out the tower reduces to equation (56). It should estimate the tower correction to the Weinberg operator and to active–sterile mixing, including the dependence on R, M_N, and M_Φ. If localization by odd bulk masses is invoked, the same calculation must use the localized eigenfunctions rather than the unperturbed trigonometric tower. Without these steps, the claims that the construction is a viable UED seesaw and that KK contributions may be ignored are unsupported.

3. **The texture in equation (56) is not the most general one allowed by the paper's own assumptions.**

   The footnote following equation (12) explicitly observes that a neutral chiral-orbifold fermion permits the bulk operator $\bar{\eta}\eta^{5c} + (\text{h.c.})$. Taking $\eta = N$ therefore allows a direct Majorana-type bulk term for the very singlet N used in equation (50), which fills the nominally zero (N, N) entry of equation (56). Boundary-localized Majorana and kinetic operators at the orbifold fixed points must also be classified. No symmetry is given that forbids these terms. Under the manuscript's own charge convention, M_N already breaks lepton number, so that convention cannot protect the omitted N Majorana operator; loop corrections will in general require every allowed local counterterm.

   The authors must state the complete operator basis compatible with the gauge symmetry and both boundary conditions, and either include its effects or provide an exact symmetry that enforces the displayed zeros. They should then show that the protecting symmetry is compatible with the N–Φ mixing and Yukawa coupling and survives radiative corrections. Otherwise equation (59) is a result for a chosen, unprotected texture rather than a prediction of the CBC construction, and its naturalness claims do not follow.

4. **Equations (57)–(58) are not a valid general flavor diagonalization with complex parameters.**

   The matrix R in equation (57) uses a transpose in its lower-left block and the manuscript evaluates RMR^{T}. This is adequate as a formal block diagonalization only when all relevant matrices are real and when kinetic normalization is tracked to the stated order. For complex Yukawa and mass matrices, a physical Majorana diagonalization is a unitary Takagi transformation: the lower-left block is controlled by a Hermitian conjugate, and complex conjugations enter the equation determining the mixing matrix. The displayed R is not unitary and therefore does not preserve canonical kinetic terms. This matters especially because the later EDM discussion requires physical CP phases.

   The authors should either state and consistently maintain a real one-generation approximation, in which case the EDM discussion must be removed, or give the unitary multi-flavor block diagonalization to second order. The active mixing matrix, non-unitarity parameter, and heavy eigenvalues should be obtained from that transformation. The number of N and Φ fields and the rank conditions needed for two observed mass splittings should also be specified. At present the model is called phenomenologically viable without reproducing the minimal flavor information contained in oscillation data.

5. **The localization overlap in equation (61) is not consistently defined.**

   For the parity in equation (51), the right-handed zero-mode profile of N must be even on $[-\pi R, \pi R]$. An odd-sign bulk mass therefore gives a profile proportional to $e^{\mu_N |y|}$ (up to the sign convention for μ_N), not $e^{\mu_N y}$ on the full covering interval. The normalization printed in equation (61) is compatible with a full-interval $|y|$ profile, whereas the exponential as written is compatible only with a half-interval convention and then differs by an orbifold normalization factor. The integration limits are omitted, so the displayed equation is internally ambiguous.

   The authors should solve the first-order zero-mode equation with the stated parity, normalize it on the same interval used in equation (21), and evaluate the overlap analytically. With the conventions displayed in equation (61), the corrected result is

   $$\frac{M_N^{\mathrm{eff}}}{M_N} = \sqrt{\frac{2\tanh(\pi R\mu_N/2)}{\pi R\mu_N}}.$$

   Thus the flat–localized overlap is not exponentially small at large $|\mu_N|R$, but it is power suppressed and need not be of order one. Therefore the conclusion $M_N^{\mathrm{eff}} \sim M_\Phi$ does not follow without a range for $\mu_N R$ and a definition of the underlying five-dimensional coefficient. The localized profiles of L and H should be treated with the same conventions when deriving M_ν.

6. **The naturalness and phenomenological claims overstate what equation (59) demonstrates.**

   For the assumed $M_\Phi \sim M_N$, equation (59) gives $m_\nu \sim M_\nu^2/M_N$. The numerical result in equation (60) therefore rests on the input $M_\nu \sim 1\,\mathrm{MeV}$, or $y_\nu \sim 10^{-5}$, rather than on an inverse-seesaw suppression by a small technically natural parameter. The paper suggests localization as an explanation, but the Lagrangian in equation (50) does not include the required odd masses, and the resulting KK theory is left for later. The text should not say that the neutrino mass is "naturally explained" until the small overlap and its radiative stability are part of the model.

   The non-unitarity estimate also needs a matrix treatment. The observable deviation is conventionally proportional to $\Theta\Theta^\dagger/2$, with flavor-dependent bounds, rather than to an undefined scalar

ϵ^2 as stated after equation (63). For the chosen parameters the order of magnitude is indeed tiny, but this is precisely because the model assumes an extremely small Dirac coupling. It is not a distinctive prediction of the CBC construction.

7. **The electron-EDM discussion is not a calculation and cannot support a claim of testability.**

   Equation (64) is presented without a diagram, loop function, flavor sum, CP-odd invariant, or supporting theory reference. An electron EDM vanishes in a real one-generation model; with several heavy neutral fermions it depends on rephasing-invariant combinations of complex mixings and on the detailed electroweak and KK spectrum. A dimensional expression proportional to $m_e\epsilon^2/M_{\mathrm{H}}^2$ does not establish either the loop order or the decoupling behavior. Moreover, the statement that 10–100 TeV scales are testable is immediately followed by an estimate of $10^{-44}\,e\,\mathrm{cm}$, about fourteen orders below the quoted bound. The generic EDM program cited as reference [20] does not make this specific model testable.

   The authors should either remove this subsection and make no EDM claim, or supply a genuine multi-flavor calculation with CP phases, Standard Model and KK diagrams, loop functions, and the applicable experimental comparison. A similar level of care is needed before claiming overall phenomenological viability; collider, electroweak, lepton-flavor, neutrinoless-double-beta, and compactification constraints cannot all be replaced by the two estimates in equations (63)–(64).

8. **The charged-scalar completion needs a gauge- and symmetry-consistent specification.**

   Equations (19)–(20) and (55) propose fields that carry a charge while obeying a conjugating boundary condition. Such a condition maps a representation to its conjugate and is not automatically compatible with an unbroken continuous gauge zero mode. One must specify the gauge group, the automorphism imposed on its gauge field at the fixed points, which gauge generators survive, and how the real zero mode can be called neutral. The scalar potential, the origin of the S expectation value, and any associated gauge boson or Goldstone mode are also absent.

   The statement after equation (55) that the coupling κ "carries lepton number" should be replaced by a spurion statement: a fixed coupling cannot transform under an exact global symmetry. If κ is a spurion, the interaction explicitly breaks the symmetry; if S breaks it spontaneously, its charge and the resulting low-energy degree of freedom must be included. Because this completion is invoked to justify the magnitude and origin of M_N, it cannot remain only a schematic operator.

9. **The scope and novelty of the free-spectrum result and the CBC-compatible interaction must be made precise.**

   The manuscript itself explains in equations (44) and the surrounding text that reference [11] contains the general masses and that the present Dirac pairs arise at $\alpha = 0$ from the degeneracy of the n and $-n$ Majorana states. A direct derivation can be valuable, but it is not clear that it constitutes a new physical result. Likewise, the paper notes that a similar interaction appeared in reference [16]. The authors should give an operator-by-operator comparison with references [11], [14], and [16], identify any genuinely new theorem or selection rule, and separate new results from pedagogical rederivations.

   If the principal novelty is instead the extended-seesaw application, then the revisions requested above are essential: the paper must show a consequence that cannot be reproduced by the standard inverse-seesaw matrix with a different charge convention. Without such a consequence, the current advance appears too incremental for a broad high-energy theory journal.

Minor comments

1. The transition from equation (7) to equation (8) should say explicitly that five-dimensional charge conjugation preserves the kinetic term by itself, while the combined charge-conjugating reflection changes the sign of the y term. The current wording first says the kinetic term is preserved and then says it is not.

2. With the standard convention $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the equality below equation (7) has the wrong sign: $i\gamma_5 = -\gamma^0\gamma^1\gamma^2\gamma^3$, not $+\gamma^0\gamma^1\gamma^2\gamma^3$.

3. Equation (35) should state $n, m \geq 1$ for the sine functions. Since $S_0 = 0$, its norm is zero rather than one. The sums in equations (24), (26), and (52) should either start the odd tower at $n = 1$ or define the vanishing term explicitly.

4. In the second-order derivation leading to equations (29)–(32), m_n should initially be introduced as a four-dimensional mass-squared separation constant. The individual fields $\psi_{\rm M}^{(n)}$ and $\psi_{\bar{\rm M}}^{(n)}$ do not obey independent first-order Dirac equations until the mixing in equation (38) has been diagonalized.

5. Equations (39)–(43) should discuss the branch of θ_n when the bulk mass m is negative. Equation (43) prints the zero-mode mass as m, whereas Figure 1 correctly labels the physical mass $|m|$.

6. The accidental $U(1)$ below equation (44) acts separately at each quadratic KK level. It should not be described in a way that suggests a local five-dimensional current. The authors should state which bulk interactions preserve KK number, which couple different levels, and which merely split the two Majorana masses within one level.

7. The relations between the five-dimensional parameters Y_ν, Y_e, κ, M_N and the four-dimensional quantities $y_\nu, y_e, M_N^{\rm eff}$ should be collected in one place, with mass dimensions and normalization factors. At present some symbols are reused before these relations are defined.

8. Equation (54) mixes four-component Majorana notation with a symmetric left-handed mass-vector notation. The third component of Ξ should be written as the independent left-handed Weyl component of $\Phi_{\rm M}$, or the four-component derivation should be displayed carefully enough to make all factors of $1/2$ unambiguous.

9. The name $M_{\rm Diag}$ in equation (58) is misleading because residual off-diagonal terms of higher order are displayed. State the truncation order and include the normalization terms required at order ϵ^2.

10. Equation (44) should specify the range of the integer n, the periodicity of α, and how the n and $-n$ states are counted at $\alpha = 0$. This would make the comparison with the paper's $n \geq 1$ Dirac tower transparent.

11. The prose should distinguish a Majorana condition, a Majorana mass parameter, and physical lepton-number violation. These notions are repeatedly treated as equivalent even though the main model is meant to demonstrate that they are not.

Recommendation

The manuscript has a sound and potentially useful free-field core, and the construction of boundary-compatible bulk interactions is worth developing. I cannot recommend publication in the present form because the advertised lepton-number interpretation has not been established, the claimed extra-dimensional model has been truncated before its characteristic KK physics is analyzed, and the flavor and EDM calculations are not yet controlled. A substantially revised manuscript could become publishable if it supplies a full symmetry/spurion analysis, correct multi-flavor diagonalization, a controlled KK reduction with specified R and cutoff, and a sharply stated result beyond the prior CBC and inverse-seesaw literature. If those additions show that the CBC produces a basis-independent low-energy or KK signature, the work would be of interest to the extra-dimensional neutrino-model community.