Referee Report

Belinfante Symmetrization from Metric-Affine Conservation Laws: Hypermomentum as the Improvement Term—The Cases of QED and QCD

arXiv:2607.11322

Recommendation: Reject in the present form, with encouragement to resubmit after substantial reconstruction. The manuscript contains a useful organizing idea and one potentially interesting new direction, namely the use of a particular nonmetricity coupling to reproduce the conformal scalar improvement. The flat-space reduction of the metric-affine identities is transparent, the final QED tensor in equation (33) has the standard form, and equations (54)–(59) recover the Callan–Coleman–Jackiw coefficient. However, the gauge-field affine lift in equations (26) and (40) breaks the internal gauge symmetry whose theories are being studied; the purported general sufficient class in equation (48) is contradicted by the manuscript's own equation (50); the lift is non-unique in precisely the manner displayed in equation (78); and the QCD calculation contains an internal sign error and omits its essential non-Abelian derivation. In addition, the headline geometric identification and the separate Maxwell and Dirac examples substantially overlap both classic metric-affine literature and reference [25]. These issues affect correctness, novelty, and the interpretation of the main result, and cannot be resolved by a limited revision.

Summary and overall assessment

The manuscript begins by reviewing metric-affine geometry and the canonical, metrical, and hypermomentum currents. Equations (12) and (13) are the metric-affine balance identities. On setting the metric and connection to their Cartesian Minkowski values, the author obtains equations (14) and (15),

$$\partial_\mu t^{\mu\nu} = 0, \qquad t^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\alpha \Delta^{\nu\mu\alpha}.$$

An "affine lift" is then used as a prescription: covariantize a flat-space theory, vary the lifted action with respect to the independent connection, return to flat space, and use equation (15) to reconstruct the metrical tensor. Section IV applies this prescription to QED and QCD. Section V asks when the flat hypermomentum gives an identically conserved improvement, proposes a class of torsion and nonmetricity couplings, constructs a scalar coupling based on $Q_\mu - q_\mu$, and derives the standard traceless scalar stress tensor. It ends with an expansion in connection-dependent couplings and an alternative direct metric variation.

There is a sound core at the level of the established Noether identity. Provided the matter equations are imposed with the stated conventions, equations (14) and (15) express the familiar relation among canonical, Hilbert, and hypermomentum currents. For the displayed scalar example, the double divergence does cancel and the value in equation (57) produces the standard tensor (59). These observations could support a focused paper about equivalence classes of metric-affine extensions and the respective roles of torsion and nonmetricity.

The present claims go considerably further. The paper treats the affine lift as a preferred geometric derivation, claims a broad nonmetricity classification, and presents QED and QCD as substantive new applications. The lift used for the gauge sectors is not compatible with internal gauge invariance, while equation (78) shows that the split into $T^{\mu\nu}$ and ${\Delta_\lambda}^{\mu\nu}$ changes under arbitrary relocalizations. The proposed classification does not account correctly for the algebraic symmetry of nonmetricity. Finally, the part most likely to be new—the scalar nonmetricity construction—is not compared adequately with the torsion construction already given in reference [25] or with the established scalar–nonmetricity literature. I therefore do not consider the current manuscript suitable for PRD or JHEP.

Major comments

1. **The gauge-field affine lift breaks internal gauge invariance, and the disputed term is exactly what produces the gauge-field hypermomentum.**

   For an Abelian gauge potential, the gauge-invariant field strength on any manifold is the exterior derivative $F = \mathrm{d}A$. It is independent of the spacetime affine connection. Equation (26) instead defines
$$\mathbf{F}_{\mu\nu} = F_{\mu\nu} + 2S_{\mu\nu}{}^{\lambda}A_{\lambda}.$$
   Under $A_\mu \mapsto A_\mu + \partial_\mu\chi$, this object transforms as
$$\mathbf{F}_{\mu\nu} \mapsto \mathbf{F}_{\mu\nu} + 2S_{\mu\nu}{}^{\lambda}\partial_\lambda\chi,$$
   and is therefore not $U(1)$ gauge invariant away from the final torsionless slice. The same problem occurs in equation (40): the Yang–Mills curvature is $G = \mathrm{d}A + g_s A \wedge A$, whereas replacing its exterior derivative by an affine curl adds a torsion term multiplying a connection that transforms inhomogeneously. The resulting $\mathbf{G}_{\mu\nu}$ is not adjoint-covariant under a local color transformation.

   This is central rather than cosmetic. The terms $2A_\lambda F^{\mu\nu}$ in equation (31) and $2A^a_\lambda G^{a\mu\nu}$ in equation (43) arise precisely from the linear torsion coupling introduced by this non-gauge-invariant covariantization. If one uses the internally gauge-invariant lift $F = \mathrm{d}A$ and $G = \mathrm{d}A + g_s A \wedge A$, the gauge-field contribution to the connection variation vanishes. The claimed derivation of the Maxwell or Yang–Mills part of the Belinfante tensor then no longer follows from the stated algorithm.

   The authors must compare these two lifts explicitly and give a principled admissibility criterion. If the affine-curl theory is intended only as an auxiliary bookkeeping extension, that status and its loss of gauge symmetry must be stated, and the claims that the resulting hypermomentum has a preferred physical meaning must be reduced accordingly. If internal gauge invariance is an admissibility condition, the present QED and QCD constructions have to be reformulated.

2. **The affine lift is non-unique, so the procedure does not by itself select "the" improvement tensor.**

   Equation (78) makes the ambiguity explicit:
$$T^{\mu\nu} \mapsto T^{\mu\nu} - \frac{1}{2}\partial_\alpha\chi^{\nu\mu\alpha}, \qquad \Delta^{\nu\mu\alpha} \mapsto \Delta^{\nu\mu\alpha} + \chi^{\nu\mu\alpha}.$$
   More generally, one may add to the lifted action local terms linear in torsion or nonmetricity that vanish when $g = \eta$ and $\Gamma = 0$. Such terms leave the original flat theory and its canonical tensor unchanged while altering both the flat hypermomentum and the metrical tensor. The scalar coupling in equation (53) illustrates exactly this freedom: it is added and tuned after the minimal lift of a scalar, for which $\nabla_\mu\phi = \partial_\mu\phi$.

   The equality in equation (5) between the textbook canonical tensor and a coframe response also requires its hypotheses to be restated for this enlarged class of lifts. Once explicit torsion and nonmetricity couplings are allowed, including metric derivatives hidden in $Q_{\alpha\mu\nu}$, a term that vanishes at the flat background can still contribute to a coframe variation there. The manuscript should prove when the flat coframe current remains the canonical current of the original flat action.

Consequently, Steps 1–4 in Section V.D are underdetermined. They do not distinguish the Belinfante tensor from other pseudogauge-related symmetric tensors unless further requirements are imposed. Locality and covariance alone are insufficient; in the gauge examples, internal gauge invariance points to a different lift. Projective behavior, Weyl or frame-rescaling behavior, boundary terms, and derivative order can likewise distinguish inequivalent extensions.

The paper should formulate the construction as a classification of equivalence classes of lifts, state the admissibility conditions, and prove what is invariant under the allowed changes. It should also separate three statements that are presently conflated: $T^{\mu\nu}$ is symmetric by its metric-variation definition (6); the double-divergence condition (45) makes its flat divergence agree with that of $t^{\mu\nu}$; and equality of integrated Poincare charges additionally requires a superpotential structure and appropriate boundary falloff. Conservation alone does not make the gauge-dependent canonical density after equation (15) the unique "proper energy content."

3. **The proposed nonmetricity sufficient class is not established and is internally contradicted by equation (50).**

   Nonmetricity obeys the algebraic identity

   $$Q_{\alpha\mu\nu} = Q_{\alpha\nu\mu}.$$

   The functional derivative with respect to $Q_{\alpha\mu\nu}$ must therefore be projected onto this symmetry. In passing from equations (46) and (47) to equation (48), the manuscript instead treats the dependence on $Q_{[\nu\mu]\alpha}$ as if it automatically made ${\Delta_\lambda}^{\mu\nu}$ antisymmetric in μ, ν. The explicit result (50) disproves that assertion. Indeed,

   $${\Delta_\lambda}^{\mu\nu} = -2\delta^\mu_\lambda \partial^\nu \phi + \delta^\nu_\lambda \partial^\mu \phi + \eta^{\mu\nu}\partial_\lambda \phi$$

   has

   $${\Delta_\lambda}^{\mu\nu} + {\Delta_\lambda}^{\nu\mu} = -\delta^\mu_\lambda \partial^\nu \phi - \delta^\nu_\lambda \partial^\mu \phi + 2\eta^{\mu\nu}\partial_\lambda \phi,$$

   which is generally nonzero. Equation (51) works for this particular current because the three second-derivative terms cancel, not because the hypermomentum has the antisymmetry claimed before equation (48).

   The failure is easy to see beyond this example. Replacing $\partial^\nu \phi$ in equation (49) by a general vector V^ν gives

   $$\partial_\mu \partial_\nu {\Delta_\lambda}^{\mu\nu} = \Box V_\lambda - \partial_\lambda (\partial \cdot V),$$

   which is not an identity. Equations (50)–(51) work because $V_\mu = \partial_\mu \phi$ is a gradient; after multiplication by ϕ, the vector remains a gradient, $\phi \partial_\mu \phi = \frac{1}{2}\partial_\mu \phi^2$.

   This distinction invalidates the blanket statement that every Lagrangian of the form (48), or every coupling involving only $Q_{[\alpha\beta]\gamma}$ in the outcome of Section V.D, is guaranteed to be an improvement. Equations (66) and (69) do not repair the problem. They require simultaneously

   $$Z^{abc} = Z^{acb}, \qquad Z^{abc} = -Z^{bac}.$$

   These two pairwise symmetries force Z to vanish: successive use of them gives $Z^{abc} = Z^{acb} = -Z^{cab} = -Z^{cba} = Z^{bca} = Z^{bac} = -Z^{abc}$. Thus the alleged algebraic sufficient class (69) is trivial, and the nonzero scalar expression proposed after it cannot possess both stated symmetries. Nontrivial examples must instead satisfy the differential condition (68), as the gradient example does. The authors should redo the constrained variation, reconcile equations (46), (48), (50), and (67)–(69), and state a correct theorem in terms of

   $$\partial_\mu \partial_\nu {\Delta_\lambda}^{\mu\nu} \equiv 0$$

modulo superpotentials. Any claimed scalar uniqueness near equation (69) must also state its restrictions: derivative order, parity, engineering dimension, dependence on ϕ, and use or exclusion of the Levi-Civita tensor.

4. **The scalar conformal construction needs a complete symmetry statement, a corrected derivation, and comparison with prior torsion and nonmetricity formulations.**

The calculation in equations (54)–(59) is the strongest part of the manuscript. On the massless scalar equation of motion, equation (57) gives the standard coefficient $(n-2)/[4(n-1)]$ in equation (59). What is new is therefore not the flat stress tensor but, at most, one auxiliary nonmetricity lift that produces it.

The claim following equation (63) that the action is conformally invariant is incomplete because the transformation of the independent affine connection is not specified. If $\Gamma^{\lambda}{}_{\mu\nu}$ is held fixed while $g_{\mu\nu} \mapsto e^{2\Omega} g_{\mu\nu}$, then one finds

$$Q_\mu - q_\mu \mapsto Q_\mu - q_\mu - 2(n-1)\partial_\mu \Omega,$$

which does make the derivative (62) covariant for a scalar of weight $(2-n)/2$. This calculation should be displayed. If the connection transforms by a frame, projective, or Weyl transformation, the conclusion changes and must be rederived. The use of Ω in equation (64) and θ in the following sentence compounds the ambiguity.

The assertion that nonmetricity is "essential" is also untenable as written. Reference [25], Section 5.1, uses a torsion-vector Weyl covariantization and obtains its equation (5.19), explicitly identified with the same conserved, symmetric, traceless scalar tensor. Independently, the usual $\xi R\phi^2$ lift produces the CCJ tensor without nonmetricity. The current construction should be compared equation by equation with the torsion construction in reference [25] and with scalar–nonmetricity formulations based on $Q_\mu - q_\mu$, including L. Järv, M. Rünkla, M. Saal, and O. Vilson, *Phys. Rev. D* **97**, 124025 (2018), arXiv:1802.00492. The manuscript should identify a property that is genuinely special to the new lift rather than claiming impossibility of known alternatives.

There are also concrete algebraic omissions in the direct route. For the action (53), equation (75) must carry the factor $\lambda\phi$ multiplying the displayed bracket; as printed it is the derivative of the unmultiplied coupling (49), yet equation (76) suddenly restores both λ and ϕ. Moreover, the direct metric variation contains an additional term $+\lambda\eta^{\mu\nu}\phi\Box\phi$ off shell. Equation (76) follows only after imposing $\Box\phi = 0$, an on-shell step not stated there. The sentence between equations (58) and (59) should read

$$\partial^\mu\partial^\nu\phi^2 = 2(\partial^\mu\phi\,\partial^\nu\phi + \phi\,\partial^\mu\partial^\nu\phi),$$

not the expression printed. Finally, the claim after equation (60) that the usual Weyl derivative gives a traceless but nonconserved tensor is central to the motivation for (62), but neither that tensor nor its divergence is shown. The two constructions should be derived side by side.

5. **The QED and QCD calculations require correction and a reproducible non-Abelian derivation.**

The quark contribution to the canonical tensor in equation (39) has the opposite sign from the electron contribution in equation (25), although the fermion kinetic terms in equations (19) and (34) have the same sign and the definition (5) is unchanged. Applying (5) to (34) gives a $+i/2$ coefficient for $\bar\psi_f\gamma^\mu\partial^\nu\psi_f - (\partial^\nu\bar\psi_f)\gamma^\mu\psi_f$, not the $-i/2$ printed in (39). With the displayed sign, the asserted passage to the positive symmetrized quark contribution in equation (44) does not follow.

For QED, equation (25) has already omitted the $\eta^{\mu\nu}\mathcal{L}_D$ term, which vanishes only after using the Dirac equations. Equation (32) is also explicitly on shell. The manuscript should state consistently at each stage whether it is using an off-shell Noether identity or an on-shell representative. Equation (31) additionally adds a term containing γ_ν to one containing γ^ν with the same free slot; the index placement must be corrected.

Most importantly, the entire non-Abelian step from equation (43) to equation (44) is replaced by the phrase "reproduces precisely." This is the only place where the QCD example could contain content beyond combining the previously treated Maxwell and Dirac sectors. The authors should display the divergence, use $(D_\mu G^{\mu\nu})^a$ explicitly, and show how all $g_s f^{abc} A^b A^c G^a$ and color-current terms combine. A more meaningful result would treat an arbitrary compact gauge group and fermion representation. As it stands, the sign error plus the missing calculation prevents verification of equation (44).

The scope should also be described accurately. Equations (34)–(44) concern the classical Yang–Mills–Dirac stress tensor. They do not address gauge fixing, ghosts, BRST currents, composite-operator renormalization or mixing, or the trace anomaly. Several references cited after equation (44) study these genuinely quantum or operator-level questions. "Perfect agreement" should be replaced by a precise comparison of the same classical tensor, or the quantum ingredients should be included.

6. **The novelty claim must be reorganized around what is actually new.**

   The manuscript itself notes below equation (15) that the flat identities appeared in the 1977 work cited as reference [23], and the canonical–metrical–hypermomentum relation is standard in the metric-affine literature cited in Section III. More directly, reference [25] already presents the flat hypermomentum as the geometric origin of Belinfante–Rosenfeld improvement and treats Maxwell and Dirac fields separately. The QED calculation in equations (19)–(33) is largely their coupled sum, while the QCD section is a direct color generalization whose decisive algebra is not displayed.

   The Introduction, abstract, title, and conclusions currently present the old identity as the main advance and give insufficient attention to the only plausible new component, the nonmetricity coupling and attempted classification in Section V. The comparison after equation (44) is particularly unhelpful: the cited papers address different questions, so elegance cannot establish superiority or novelty.

   A viable new manuscript would sharply separate (i) the established MAG Noether identity, (ii) the previously published Maxwell, Dirac, and torsion-scalar lifts, and (iii) a corrected new result about admissible nonmetricity lifts. It could then analyze whether internal gauge invariance, projective symmetry, or Weyl/frame covariance selects representatives within the pseudogauge family. Without such a refocusing and a valid classification theorem, the present incremental examples do not meet the novelty threshold of a high-energy theory journal.

Minor and presentation comments

1. The irreducible split in equation (8) is inconsistent with the definitions (9)–(11): the spin piece is denoted σ in (9) and the shear piece Σ in (11), whereas (8) writes Σ in the first slot and an undefined $\widehat{\Delta}$ in the final slot. The decomposition and all index placements should be corrected.

2. The footnote attached to equation (5) writes the tetrad relation with the tangent index a repeated more than twice. It should be $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b g_{ab}$.

3. The corollary in Section V.D attributes the Belinfante relation to equation (13). The relation between $t^{\mu\nu}$, $T^{\mu\nu}$, and $\Delta_\lambda{}^{\mu\nu}$ descends from equation (12); equation (13) gives the balance equation that becomes conservation of the canonical tensor.

4. Section V.C refers to “equation (18a),” but equation (18) has no subequations. The appended letter is generated in the source rather than by an actual label.

5. Equation (73) writes $T^{\mu\nu}$ while its first term is $g_{\mu\nu}\mathcal{L}_M$ and the remaining derivatives also carry lower free indices. Either the equation should define $T_{\mu\nu}$ or every term should be raised consistently before the flat limit.

6. Direct antisymmetrization of equations (15) and (67) gives a plus sign in front of $2\partial_\lambda Z^{\lambda[\nu\mu]}$ in equation (70), not the printed minus sign. The term vanishes under the symmetry assumed in the next sentence, so equation (71) is unaffected, but the algebra should be corrected.

7. The constant λ in equation (53) is dimensionless for a canonically normalized scalar in n dimensions, not dimensional as stated below that equation.

8. The condition in equation (45) ensures preservation of the local conservation equation. Calling the associated term an improvement with the same charges additionally requires boundary or falloff assumptions; these should be stated.

9. The QED discussion says that the lift is Riemann–Cartan because of spinors, while the general definition and final algorithm advertise an arbitrary metric-affine connection. The independent variables and allowed connection variations for fermions should be specified. If only the Lorentz projection of the spin connection couples, this should be written explicitly.

10. The Introduction says that the Hilbert construction “breaks down” for a Dirac field. This is inaccurate: fermions are coupled through a tetrad and spin connection, and tetrad variation is the standard gravitational definition of their stress tensor. The need for a tetrad is not a failure of the variational construction.

11. Equation (42) drops the flavor labels inside the sum over f. This is not harmless if different masses are intended.

12. The affiliation contains a corrupted dash in the rendered manuscript. There are also numerous language errors, including “of-shell” for “off-shell,” “capable” for “sufficient,” “realtion,” “energy-moemntum,” “repsectively,” and “in principal” for “in principle.” A careful copy edit is needed after the technical revision.

Recommendation

The manuscript’s clear reduction of the metric-affine identities and the scalar calculation in equations (54)–(59) show that there is useful material here. Nevertheless, the present gauge examples rely on a curved-space extension that violates their internal gauge symmetries, the purported nonmetricity class is not supported by its constrained index algebra, and the main conceptual result substantially predates this work. The authors would need to reconstruct the paper around equivalence classes of affine lifts, correct and prove the nonmetricity classification, compare torsion and nonmetricity covariantizations, and provide the missing non-Abelian calculation. I therefore recommend **rejection in the present form**, while encouraging submission of a substantially refocused manuscript if these issues are resolved.