Referee Report

Brouwer Degree of Thermodynamic Multicritical Points in Black Holes

arXiv:2607.11394

Recommendation: Reject in the present form; a fundamentally reconstructed manuscript could be considered as a new submission. The idea of following critical points as signed zeros of a response-function map is potentially useful, and the manuscript contains a clear elementary discussion of local degree. However, the main physical and numerical claims do not survive direct checks. The Power–Maxwell equations and parameters printed in the paper give three physical critical points with total degree -1, rather than the asserted $2 \to 4 \to 6$ sequence with total degree -2. The six-dimensional Gauss–Bonnet mass and temperature are not obtained from the displayed metric, and the pressure–AdS-radius conversion is four-dimensional rather than six-dimensional. The Euler–Heisenberg continuation reverses the charges of the two branches and makes a false claim about their eventual disappearance. More fundamentally, no multiphase coexistence or multicritical point is demonstrated, no domain is supplied for the claimed global Brouwer degree, and closely related Brouwer-degree classifications of black-hole critical points predate this work. These issues affect the central results, novelty, and interpretation and cannot be repaired by a limited revision.

Summary and overall assessment

The paper defines

$$\phi(r,P) = \left(C_P^{-1}, \partial_r C_P^{-1}\right)$$

and identifies a simultaneous zero with an inflection point of the equation of state. Section 2 reviews the winding of the normalized field, the sign of the Jacobian determinant at a regular zero, and a classification of the Jacobian eigenvalues as saddles, nodes, or foci. Section 3 applies the construction to RN–AdS and obtains one defect of degree -1. Section 4 finds two Euler–Heisenberg defects of opposite degree and attempts to follow their motion as the charge varies. Section 5 reports pair annihilation in six-dimensional charged Gauss–Bonnet–AdS and pair creation in a nonlinear Power–Maxwell model. The conclusion assigns total charges -1, 0, -1, and -2 to the four models.

At the level of a regular equation of state, the local construction is mathematically straightforward. Writing $g = C_P^{-1}$, the proposed map is the first jet (g, g_r). At a regular zero,

$$J = \begin{pmatrix} 0 & g_P \\ g_{rr} & g_{rP} \end{pmatrix}, \qquad \det J = -g_P g_{rr}.$$

Thus a local sign can indeed be assigned, and a generic creation or annihilation of two zeros has opposite signs. The paper's difficulty is not this elementary local statement, but the attempt to elevate it to a new global invariant and a physical dynamics without the necessary hypotheses, together with substantial algebraic and numerical errors in the examples. The manuscript also does not establish that the zeros being classified correspond to genuine multicritical coexistence rather than several ordinary critical endpoints.

Major comments

1. **Equations (1) and (2) are equivalent only on a restricted regular domain, which is never defined.**

   At fixed pressure and other conserved charges,

   $$C_P = \frac{\partial M}{\partial T} = \frac{M_r}{T_r}, \qquad \frac{1}{C_P} = \frac{T_r}{M_r}.$$

   For an ordinary black hole satisfying the first law, $M_r = TS_r$ at fixed $(P, Q, \ldots)$. Consequently the first component of Eq. (2) vanishes with T_r, and its r derivative vanishes with T_{rr}, only where $TS_r \neq 0$. Equivalence to the constant-temperature conditions in Eq. (1) also requires $T_P \neq 0$. These conditions can fail at extremality, at singular points of the chosen ensemble, or at boundaries on which the heat capacity is undefined. In addition, C_P^{-1} has poles where $C_P = 0$, so the map need not be continuous on a region containing all claimed defects.

   The authors should state the thermodynamic ensemble, define the physical domain in (r, P), list all excluded singular sets, and prove the equivalence of Eqs. (1) and (2) on that domain. This is essential for any use of Brouwer degree, which applies to a continuous map on a specified domain with the target value absent from the image of the boundary. Merely finding zeros of rational expressions is not sufficient.

2. **The asserted global conservation law is not established and is contradicted by boundary crossings in the examples.**

   Equation (22) defines a sum of local signs, but it does not define a global degree $\deg(\boldsymbol{\phi}, \Omega, 0)$ because no bounded region Ω or boundary condition is given. Under variation of a parameter, the sum is conserved only if the map remains continuous and nonzero on a fixed boundary; defects may otherwise enter or leave through $r = 0$, $P = 0$, $T = 0$, infinity, or another physical boundary. Pair creation and annihilation with zero net index is a standard local consequence of degree theory, not by itself evidence for a model-specific conserved thermodynamic charge.

   The issue is visible in Table 3. RN–AdS is listed as having both one and zero critical points while retaining $W_{\text{tot}} = -1$. This is incompatible with the paper's own definition $W_{\text{tot}} = \sum_i w_i$: an empty set has sum zero unless a boundary or point at infinity supplies an additional contribution. The Euler–Heisenberg positive-degree branch also crosses $P = 0$ as Q increases, changing the sum within the physical AdS region. The authors must choose a fixed compactified domain, compute its boundary degree, and track boundary flux. Without that analysis, comparing the numbers -1, 0, and -2 across different black-hole systems has no invariant meaning.

3. **The manuscript does not demonstrate multicriticality as defined in its own Introduction.**

   Equation (1) detects an ordinary second-order endpoint of a first-order coexistence line. Several isolated solutions (r_i, P_i, T_i) at different pressures and temperatures are several critical points; they are not automatically a multicritical point at which more than three phases coexist. Establishing an n-tuple point requires the relevant branches to have equal Gibbs free energy at a common (P, T), or an equivalent construction of overlapping swallowtails and coexistence lines. No Gibbs free energy, equal-area condition, coexistence curve, or phase diagram appears in the manuscript.

   This distinction is decisive in the Power–Maxwell example. The numbers in Eq. (71) are the benchmark for three *separated* swallowtails in M. Tavakoli, J. Wu, and R. B. Mann, "Multi-critical Points in Black Hole Phase Transitions," arXiv:2207.03505, JHEP 12 (2022) 117, together with its erratum JHEP 12 (2023) 012. They are not the parameter set for the quadruple point in that work;

the latter uses $Q = 6.75112$ and different higher-order couplings. The present paper should cite the original article and erratum, distinguish extrema at fixed pressure from critical endpoints obtained by varying pressure, and demonstrate actual phase coexistence. As written, the examples concern multiple ordinary critical points, not the topology of a multicritical point.

4. **The Power–Maxwell result in Table 2 and Eqs. (72)–(73) is incompatible with Eqs. (61)–(71).**

   This can be checked without any unpublished data. Differentiating the temperature in Eq. (63), imposing $T_r = T_{rr} = 0$, and setting $y = r_+^2$ gives the critical-radius equation

$$0 = 2y^{13} - 12Q^2y^{12} - 140b_5Qy^{10} - 396b_9Qy^8 - 780b_{13}Qy^6 - 1292b_{17}Qy^4 - 1932b_{21}Qy^2 - 2700b_{25}Q.$$

   Pressure is then fixed by $T_r = 0$. Using exactly the values in Eqs. (66)–(71) gives three positive, physical solutions,

r_+	P	T
7.51330	1.12338×10^{-4}	7.10217×10^{-3}
9.83281	5.56461×10^{-5}	6.14848×10^{-3}
14.75152	8.21011×10^{-5}	6.77156×10^{-3}.

   Their degrees, ordered by radius, are $(-1, +1, -1)$, so the total is -1. This is also the pattern expected from the three distinct critical endpoints associated with the three separated swallowtails of the source benchmark. The polynomial has positive leading coefficient and, for the displayed parameters, negative constant term; simple positive roots therefore occur in odd number. The claimed configurations with two, four, and six zeros cannot follow from the printed equations.

   Table 2 gives no charge intervals, critical coordinates, pressures, temperatures, determinants, or eigenvalues, despite the text saying that locations are summarized there. The authors must provide the actual continuation data and code, explain what was counted, and reconcile the computation with the analytic equation above. It appears likely that the six extrema of $T(r_+)$ at one fixed pressure have been confused with zeros satisfying both criticality conditions. Until this section is recalculated, the headline transition $2 \to 4 \to 6$ and $W_{\text{tot}} = -2$ are unsupported.

5. **The six-dimensional Gauss–Bonnet thermodynamics is internally inconsistent before any topology is computed.**

   Solving the horizon equation from the metric function in Eq. (44), with $\widetilde{\alpha} = 6\alpha$ and the standard six-dimensional relation $\Lambda = -10/l^2$, gives

$$M_{(44)} = \frac{1}{4\pi}\left(r_+^3 + 6\alpha r_+ + \frac{r_+^5}{l^2}\right) + \frac{q^2}{96\pi r_+^3}.$$

   Equation (45), specialized to $D = 6$, instead gives

$$M_{(45)} = \frac{3}{16\pi}\left(r_+^3 + 6\alpha r_+ + \frac{r_+^5}{l^2}\right) + \frac{q^2}{32\pi r_+^3}.$$

   These expressions are not related by a common normalization and cannot both follow from the same metric. Equation (46) inherits the mismatch. Moreover, from $P = -\Lambda/(8\pi)$ and $\Lambda = -(D-1)(D-2)/(2l^2)$ one obtains

$$P = \frac{5}{4\pi l^2}$$

in six dimensions, not $3/(8\pi l^2)$ as stated below Eq. (46). The latter is the four-dimensional formula. Direct differentiation of Eq. (44) gives

$$T_{(44)} = \frac{-q^2 + 48\alpha r_+^4 + 24 r_+^6 + 32\pi P r_+^8}{32\pi r_+^5 (r_+^2 + 12\alpha)}.$$

This is not Eq. (46). The reported collision therefore does not establish a property of the black hole defined by Eq. (44): at the point quoted in Eq. (53), the correct temperature gives $T_r = -0.25944$ and $T_{rr} = 0.72259$, rather than zero. There are also two different couplings: the paragraph and Table 1 use $\alpha = 0.00805$, while Fig. 5, Eq. (53), and the subsequent text use $\alpha = 0.00815$. The metric, mass, temperature, pressure convention, and charge normalization must be rederived consistently and the entire zero set recomputed.

6. **The Euler–Heisenberg continuation reverses the two charge labels and makes false claims about the physical branches.**

   The numerator of Eq. (34), together with Eq. (35), allows an analytic check. With $x = r_c^2$, the two zero conditions reduce to

   $$x^3 - 6Q^2x^2 + 7aQ^4 = 0, \qquad P_c = \frac{3x - 6Q^2}{32\pi x^2}.$$

   For $a = 0.04$ and $Q = 0.1$, the two physical points are

   $$(r_c, P_c) = (0.176515, 0.342975), \qquad (0.218374, 0.363326).$$

   The first is the positive-degree focus whose eigenvalues are precisely those printed in Eq. (40); the second is the negative-degree saddle. Figure 2 assigns them accordingly. Figures 3 and 7 then interchange the labels: their blue $w = +1$ trajectory is the large-radius saddle branch, while their red $w = -1$ trajectory is the small-radius focus branch.

   The error propagates into the prose. At $Q = 0.1$, implicit differentiation gives

	$\mathrm{d}r_c/\mathrm{d}Q$	$\mathrm{d}P_c/\mathrm{d}Q$
$w = +1$	-0.15404	-12.77020
$w = -1$	3.34963	$-8.34367.$

   Thus the positive-degree branch initially moves inward, not outward; the negative-degree branch has the much larger outward radial velocity; and the magnitude of the pressure velocity is larger, not smaller, on the positive-degree branch. These statements directly contradict Section 4.1 and the discussion after Eq. (81).

   The exact cubic also determines branch existence. The pair is created at $Q = \sqrt{7a/32} = 0.0935414$. The positive-degree focus reaches the physical boundary $P = 0$ at $Q = \sqrt{7a/16} = 0.132288$, whereas the negative-degree saddle remains at positive pressure for every larger Q. Hence the Appendix claim that both defects eventually disappear and that no positive real critical point remains is false. Table 3's Euler–Heisenberg entry "$2, 0$" is not the physical sequence under increasing Q; within $P > 0$ it is $0 \to 2 \to 1$. This example must be corrected before it can support any conservation claim.

7. **The claimed stability, flow, and Lee–Yang interpretation is not invariant or physically defined.**

   The Jacobian in Eqs. (8)–(9) is the derivative of a map from thermodynamic coordinates to two response functions. No dynamical equations $\dot{r} = \phi_1$, $\dot{P} = \phi_2$ are derived. Therefore negative real parts of its eigenvalues do not establish thermodynamic or dynamical stability. Moreover, C_P^{-1} and $\partial_r C_P^{-1}$ have different physical dimensions, so the Euclidean norm in Eq. (4) is not defined until arbitrary relative scales are chosen; Eq. (5) likewise cannot use a single dimensionful ϵ for both r and P. A dimensionless rescaling can be introduced for computing a winding, but its choice must be stated.

The degree is robust under an orientation-preserving rescaling of the target components, but the eigenvalue classification is not. For example, multiplying only the second component of $\boldsymbol{\phi}$ by a positive constant leaves every zero and degree unchanged, yet it can turn the Euler–Heisenberg complex pair in Eq. (40) into two real negative eigenvalues. A simple change of units can therefore turn the alleged focus into a node.

For the same reason, the resemblance asserted below Fig. 4 to Lee–Yang zeros has no demonstrated content. Lee–Yang zeros are zeros of a partition function in a complexified thermodynamic variable; the plotted numbers are eigenvalues of an arbitrarily normalized local Jacobian. The authors should either derive a physical evolution law and an invariant relation to response stability, or remove the node/focus stability language and the Lee–Yang analogy. The sign of the determinant alone is the coordinate-level local degree established by the present construction.

8. **The parameter-space distances and speeds in Eqs. (42) and (49) are dimensionally and geometrically undefined.**

   The horizon radius has dimension L, pressure has dimension L^{-2}, and charge has its own model-dependent dimension. Consequently

   $$(r_i - r_j)^2 + (P_i - P_j)^2$$

   adds quantities of dimensions L^2 and L^{-4}. Even after numerical units are chosen, the Euclidean norm changes under a harmless rescaling of either axis. The position and height of the maximum in $\mathrm{d}D/\mathrm{d}Q$ in Fig. 3 are therefore coordinate artifacts, as are comparisons of the magnitude of "defect velocities." Only the simultaneous statement $r_i = r_j$ and $P_i = P_j$ is independent of such a norm.

   If a distance is physically important, the authors must introduce dimensionless variables and justify a metric on the thermodynamic parameter space. Otherwise they should plot the two coordinate differences separately and use the vanishing determinant to diagnose a fold. Figure 6 must also give three distinct charge values: its caption assigns $q = 0.00205$ to the before, during, and after panels even though Eq. (53) places the collision at $q_c = 0.00218216$.

9. **The novelty claim does not engage the closest prior Brouwer-degree construction.**

   N.-C. Bai, L. Li, and J. Tao, "Topology of black hole thermodynamics in Lovelock gravity," arXiv:2208.10177, already use the spinodal curve to assign a Brouwer degree to black-hole thermodynamic critical points, sum local charges into a total charge, derive it from boundary behavior, and discuss the annihilation of adjacent critical points with opposite signs. This is substantially closer to the present proposal than the broad thermodynamic-topology references cited in the Introduction. In the regular region, the current two-dimensional map (g, g_r) is a jet embedding of the same criticality data; its determinant sign reduces to a sign built from the spinodal derivatives.

   The manuscript must cite and compare this work and subsequent applications. It should identify precisely what the two-dimensional embedding proves that the earlier one-dimensional Brouwer degree does not. Parameter continuation and a non-invariant Jacobian eigenvalue plot are not, by themselves, a substantial advance. A publishable reconstruction would need a correct and invariant theorem specifically tied to genuine multiphase coexistence, together with examples that reveal new physics rather than repackaging the generic pair rule.

10. **The numerical results are not reproducible at the level required for a journal article.**

    Apart from one Euler–Heisenberg eigenvalue pair and the Gauss–Bonnet collision point, the paper gives no coordinates, parameter intervals, contour sizes, determinants, numerical tolerances, or continuation algorithm. Table 1 changes α between text and figures. Table 2 contains only qualitative signs and no locations despite the sentence claiming otherwise. The Power–Maxwell charge is simultaneously said to be fixed at $Q = 6.623$ and varied through unspecified regimes. The dimensionful couplings in Eq. (71) are quoted as bare numbers without a scale convention. Figure 6 omits the three parameter values needed to distinguish its panels.

    The revised work would need machine-readable tables or code for every branch, explicit physicality cuts ($r_+ > 0, P > 0, T > 0$), and substitution residuals for both components of $\boldsymbol{\phi}$. It should show Gibbs free energies and phase diagrams wherever multicriticality is claimed. Given the analytic contradictions above, this is not merely a request for supplementary presentation; the calculations must first be redone.

Minor and presentation comments

1. The bibliography contains no entries for either `Cai:2013qga` or `PowerMaxwell`; both citations therefore render as undefined. The latter omission conceals the provenance and erratum of Eq. (71).
2. Equation (64) is an empty numbered line created by the final line break in the preceding alignment and should be removed.
3. The paper alternates between calling Q magnetic and electric in the Euler–Heisenberg section. The ensemble and charge convention should be consistent.
4. In the statement $C_P = \mathrm{d}M/\mathrm{d}T$ below Eq. (65), replace total derivatives by a partial derivative and list the variables held fixed, including P, Q, and all nonlinear couplings.
5. The caption of Fig. 3 describes only panel (a), not panel (b). The vertical axis of panel (b) should also carry the dimensional normalization used for D.
6. The caption of Fig. 5 calls its left panel a trajectory plot, although it plots three pairwise distances. It should identify which critical points carry which charges and give the numerical parameter range.
7. Figure 1 has the typographical error “inding number.” “Gauss–Bonnet” is misspelled in Fig. 6, and “minimum eigenvalue” should not be split into “minimum eigen value.”
8. The three empty `\label{}` commands in Fig. 6 generate multiply defined labels and should be deleted or replaced with unique labels.
9. The statement after Eq. (52) needs a full stop rather than a comma. More generally, the manuscript requires careful copy-editing for repeated words, subject–verb agreement, punctuation, and phrases such as “more then,” “Ffixing,” and “Whether this similarities.”
10. The action in Eq. (58) should state the normalization of Newton’s constant and the sign convention for F^2. The dimensions in Eq. (71) should then be supplied in the same convention.
11. The RN–AdS benchmark should print (r_c, P_c, T_c) and the Jacobian, rather than only inequalities, so that Eq. (23)–(30) can serve as an actual normalization check. The final literal code-fence characters after `\end{document}` should also be removed from the manuscript source.