# Referee Report

*Superconformal mechanics from $\mathcal{N}$-extended Euler–Calogero–Moser and Calogero models*

arXiv:2607.11468

**Recommendation: Reject in the present form, with encouragement to resubmit after a complete algebraic reconstruction.** The manuscript contains a potentially interesting observation: the relative two-particle Euler–Calogero–Moser (ECM) system appears to possess a second set of supercharges, suggesting an enhancement of the manifest $OSp(\mathcal{N}|2)$ symmetry to $SU(1,1|\mathcal{N})$. That point could form the basis of a useful specialist paper. However, two central current rewritings used to construct the advertised spin extensions are not algebraic identities. Equation (3.7) fails in the minimal $\mathcal{N}=2$ case, and equation (4.13) has a factor-of-two error in the same test. In addition, the bosonic current in equation (4.15) has the wrong sign to obey the stated $SO(\mathcal{N})$ algebra. Consequently, neither of the spin-extended constructions is established by the displayed formulae. The center-of-mass reductions also contain inconsistent normalizations, and the proposed reductions are substitutions rather than demonstrated Dirac reductions. These problems affect most of the claimed new results and cannot be resolved by editorial corrections alone.

## Summary and overall assessment

Section 2 recalls the arbitrary-even-$\mathcal{N}$ supersymmetric ECM system and specializes it to two particles. Equations (2.12)–(2.20) separate center-of-mass and relative variables, while equations (2.21)–(2.24) exhibit an $OSp(\mathcal{N}|2)$ conformal algebra for the relative mechanics. Section 3.1 introduces the additional charges $q^a, \bar{q}_a$ in equation (3.1), their conformal partners, and an enlarged set of fermion bilinears. These objects are intended to supply the $u(\mathcal{N})$ $R$-symmetry and hence the hidden $SU(1,1|\mathcal{N})$ superconformal symmetry.

The remainder of Section 3 attempts to add bosonic spin variables. At $g=0$, equation (3.7) first rewrites the original charges in terms of fermionic $R$-currents. Equations (3.8)–(3.13) then add a harmonic realization of bosonic currents and state a new Hamiltonian. Equations (3.14)–(3.18) are said to reduce this enlarged system to a previously known $SU(1,1|\mathcal{N})$ model. Section 3.3 reorganizes the currents in a $U(\mathcal{N}/2)\times U(\mathcal{N}/2)$ and Grassmannian basis.

Section 4 starts from the authors' earlier matrix-fermion Calogero construction. It identifies an $OSp(\mathcal{N}|2)$ relative sector in equations (4.8)–(4.14), adds bosonic $SO(\mathcal{N})$ currents in equations (4.15)–(4.18), and finally asserts a reduction chain from this model to the ECM and standard superconformal systems.

There is a sound-looking core beneath the later difficulties. In the $\mathcal{N}=2$ specialization, the reduced brackets (2.18), composite $\Pi$ in (2.16), and charges (2.19) reproduce the Hamiltonian (2.20). The hidden charges (3.1) yield the same Hamiltonian, and their mixed brackets with the original Poincare charges vanish in this test. The fermionic currents (2.23) also have the expected $u(\mathcal{N}/2)$ plus antisymmetric completion. These checks support the basic relative $OSp(\mathcal{N}|2)$ construction and motivate a corrected proof of the enhancement; they do not rescue the current rewritings or spin extensions beginning at equation (3.7).

The topic is worthwhile. A derivation of hidden enhanced symmetry directly from the two-body ECM model, valid for general even $\mathcal{N}$, would be of interest to the superconformal-mechanics community. Extra-fermion parent systems could also clarify why current-valued constructions become difficult at high supersymmetry. The present manuscript does not yet establish those broader claims. The unextended two-body reductions are closely tied to the authors' earlier models, the reduced equation (3.18) is explicitly identified with a known construction, and the genuinely new spin extensions fail direct low-$\mathcal{N}$ consistency checks. A resubmission would need to rederive the charge formulae, verify complete closure and Hermiticity, and sharply separate the surviving new result from prior work.

## Major comments

1. **The center-of-mass reductions in Sections 2 and 4 have inconsistent normalizations.**

   With the canonical variables of equation (2.12),

$$p_1 = P + p, \qquad p_2 = P - p,$$

   and therefore the kinetic part of equation (2.1) is

$$\frac{1}{2}(p_1^2 + p_2^2) = P^2 + p^2,$$

   not $\frac{1}{2}P^2 + \frac{1}{2}p^2$ as printed in equation (2.13). The interaction creates a second inconsistency. For two particles, the ordered sum in equation (2.1) gives $g^2/x^2$, whereas the fermion-free limit of equation (2.20) gives $g^2/(2x^2)$. Thus equation (2.13) is neither the direct specialization of (2.1) nor twice the relative Hamiltonian (2.20).

   The charge algebra reveals what seems to be an unstated rescaling. The exact split of equation (2.7) is the one in equation (2.15), but the sector brackets are written with $-4\mathrm{i}H_{\text{sector}}$ rather than the $-2\mathrm{i}H$ convention of equation (2.6). The definitions are consistent only if the parent Hamiltonian is identified as twice the sum of the displayed free and relative Hamiltonians. This must be stated and implemented uniformly; at present the bosonic equation (2.13), the supersymmetric equation (2.20), and the parent normalization do not agree.

   The same issue recurs more transparently in Section 4. From equation (4.5),

$$p_1\rho_{11}^a + p_2\rho_{22}^a = 2P\Psi^a + p(\rho_{11}^a - \rho_{22}^a).$$

   Consequently the sum of equations (4.6) and (4.8) is one half of the kinetic part of the original charge (4.3), although the text describes it as the separated form of that charge. Equation (4.6) also uses $\{Q_{\text{free}}, \bar{Q}_{\text{free}}\} = -\mathrm{i}H_{\text{free}}$, rather than the convention in (4.3). The authors should redo both reductions from the canonical transformations, state every rescaling of $Q$, $H$, and time, and then recheck all conformal brackets. These factors also enter the claimed reduction chain at the end of Section 4.

2. **Equation (3.7), which is the foundation of the ECM spin extension, is not an identity.**

   This can be seen without evaluating any Poisson bracket. Take $\mathcal{N} = 2$, so there is one complex index, and suppress it. From equations (2.16) and (3.5),

$$\Pi = \operatorname{Tr} U_f = \frac{\mathrm{i}}{2}(\xi\bar{\psi} - \psi\bar{\xi}), \qquad \Pi\xi = \frac{\mathrm{i}}{2}\psi\xi\bar{\xi}.$$

   The original charge in equation (3.6) is therefore

$$Q = p\psi - \frac{\mathrm{i}}{2x}\psi\xi\bar{\xi}.$$

For the right-hand side of equation (3.7), $T_f = 0$ in this one-index case, while direct Grassmann reordering gives

$$(\operatorname{Tr} U_f)\xi + (U_f)^1{}_1\xi - \mathrm{i}(V_f)^1{}_1\psi - (W_f)^{11}\bar{\xi} = \frac{3\mathrm{i}}{2}\psi\xi\bar{\xi}.$$

Thus equation (3.7) instead yields $Q = p\psi + 3\mathrm{i}\psi\xi\bar{\xi}/(2x)$. The difference is nonzero and cannot be attributed to an overall convention.

Setting all bosonic currents to zero in equation (3.10) returns precisely this faulty current expression, rather than equation (3.6). Hence equation (3.10) is not an extension of the stated ECM charge, and the Hamiltonian and Casimir formulae (3.12)–(3.13) do not follow. The authors must rederive the fermionic current decomposition directly from equations (2.16) and (3.6), test it first for $\mathcal{N} = 2$ and $\mathcal{N} = 4$, and only then add bosonic currents. Full symbolic closure of the corrected charges should be supplied, at least in an ancillary notebook or computer-algebra file, because the error propagates through the principal construction of Section 3.2.

The Grassmannian rewriting does not repair this problem. Direct substitution of equation (3.21) shows that the fermionic bilinear $\chi^a\zeta^\alpha/2$ corresponds to $(T_f + W_f)/4$, not to $(T_f - W_f)/4$ as used in equations (3.19) and (3.23). In the one-index, zero-bosonic-current case, the definition $\mathcal{Q} = (Q + \mathrm{i}q)/2$ gives

$$\mathcal{Q} = p\chi - \frac{\mathrm{i}}{x}\chi\zeta\bar{\zeta},$$

whereas equation (3.25) gives the same cubic monomial with coefficient $-3\mathrm{i}/x$. Thus equations (3.19)–(3.25) need a fresh derivation rather than an index correction alone.

3. **The hidden $SU(1,1|\mathcal{N})$ claim in Section 3.1 is promising but is not demonstrated as a complete superalgebra.**

   Equations (3.1)–(3.5) give a second set of charges and selected brackets with conformal supercharges. They do not display all mixed Poincare brackets among $Q, \bar{Q}, q, \bar{q}$, the action of every $u(\mathcal{N})$ generator on the odd charges, or the complete brackets with $H, D, K$. The constant $g$ appears additively with $\operatorname{Tr} U_f$ in equation (3.4), but its status as a coupling, a central term, or a shift of the $u(1)$ generator is not explained. The exceptional $\mathcal{N} = 2$ case, for which the relation between $su(1,1|2)$, its projective quotient, and possible central extensions matters, also deserves an explicit statement.

   This section may contain the principal result that survives the later problems. It should therefore be made self-contained: collect the generators in a standard $su(1,1|\mathcal{N})$ basis, give the complete nonzero brackets, state whether $\mathcal{N}$ initially counts real Poincare supersymmetries, and explain why the new set constitutes $2\mathcal{N}$ supersymmetries in that convention. Equation (3.2) currently refers to equation (2.7), which is a parent supercharge rather than the relative Hamiltonian (2.20). A direct low-$\mathcal{N}$ closure check should accompany the corrected general proof.

4. **Equations (3.14)–(3.18) do not define a Hamiltonian reduction.**

   The substitutions in equation (3.15) impose the fermionic constraints

   $$\phi^a = \psi^a + \mathrm{i}\xi^a = 0, \qquad \bar{\phi}_a = \bar{\psi}_a - \mathrm{i}\bar{\xi}_a = 0.$$

   Using equation (2.18), their constraint bracket is

   $$\{\phi^a, \bar{\phi}_b\} = -4\mathrm{i}\,\delta^a_b.$$

   They are therefore second-class constraints. The induced Dirac bracket is not the original bracket with the variables simply replaced; in particular, the normalization of the surviving fermionic bracket changes. Equation (3.14) checks only a post-substitution time derivative. It neither constructs the Dirac bracket nor shows preservation of all constraints, including $w^a = \bar{w}_a = 0$.

There is again a direct rank-one contradiction. For $\mathcal{N} = 2$, let $B = v\bar{v}$ and $F = \psi\bar{\psi}$. The constraint surface (3.15) gives $\{\psi, \bar{\psi}\}_D = -\mathrm{i}$, and equation (3.16) becomes

$$Q_{\text{red}} = (p - \mathrm{i}B/x)\psi, \qquad \bar{Q}_{\text{red}} = (p + \mathrm{i}B/x)\bar{\psi}.$$

Their Dirac bracket requires

$$H_{\text{red}} = \frac{1}{2}p^2 + \frac{B^2}{2x^2} - \frac{BF}{x^2}.$$

Equation (3.18) instead gives the coefficient $-3BF/x^2$. It therefore does not follow from the displayed reduced charges and, as printed, cannot coincide with the cited Kozyrev Hamiltonian.

The Calogero reduction in equation (4.19) has the same problem. The constraints $\rho^a_{21} + \rho^a_{12} = 0$ and their barred partners have a nonzero bracket, so a second-class analysis is again required. In addition, the manuscript gives no complete map from the remaining diagonal and off-diagonal matrix fermions, spin variables, $g$, and $\hat{g}$ to the variables of equation (3.25), and then to equation (3.18). To support the claimed chain, the authors should state the constraint matrix, its inverse, the reduced brackets, all invariant constraint conditions, and the explicit field and parameter maps. Equality after a formal substitution is not sufficient.

5. **The Calogero current rewriting and bosonic $R$-currents fail direct consistency checks.**

   Equation (4.13) is not equivalent to equation (4.8). Again take $\mathcal{N} = 2$ and write

$$A = \rho_{11}, \quad D = \rho_{22}, \quad b = \rho_{12}, \quad c = \rho_{21}.$$

   For one complex index, $T_f = 0$, $\Pi_{11} = Z_f$, and $\Pi_{22} = -Z_f$. Expanding the cubic part of equation (4.8) and using the definitions (4.1) and (4.14) gives, inside the common factor $\mathrm{i}/(2x)$,

$$-V_f(A - D) + Z_f(b + c).$$

   Equation (4.13) instead gives $-V_f(A - D) + 2Z_f(b + c)$, because its $Z_f$ term has coefficient $\mathrm{i}/x$. The coefficient must be $\mathrm{i}/(2x)$ for this elementary special case to agree. All subsequent formulae built from (4.13) must be rechecked after this correction.

   There is an independent sign error in equation (4.15). Its definition of $\bar{T}_{b,ab}$ is the negative of the definition in equation (3.9), even though both use the same brackets (3.8). For two distinct indices, direct evaluation gives

$$\{T_b^{12}, \bar{T}_{b,12}\} = -\mathrm{i}\big((V_b)^1{}_1 + (V_b)^2{}_2\big),$$

   whereas equation (2.24) requires the positive sign. The currents in (4.15) thus do not realize the asserted $SO(\mathcal{N})$ algebra, so the added terms in (4.16) cannot yield the claimed $OSp(\mathcal{N}|2)$ closure as written.

6. **The proposed Calogero spin extension is not a reproducible or demonstrably Hermitian model.**

   Beyond the preceding algebraic failures, equations (4.16)–(4.18) contain undefined or inconsistent quantities. The parameter $\hat{g}$ is introduced without its meaning, allowed values, or relation to $g$. Barred matrix fermions are assigned an upper $a$ index in part of equation (4.16), contrary to the conventions of Section 4. The last line of equation (4.18) contains an undefined $\rho^a$; its pattern suggests that $\eta^a$ may have been intended, but this cannot be inferred in a referee check. Several lower indices on unbarred $v, w$ and upper indices on barred $\eta$ are also unexplained.

More importantly, the nonlinear conjugation rule stated before equation (4.5) depends on $g + \Pi_{ii}$. The new charges shift the interaction by $Z_b$ with opposite signs in the two factors of equation (4.18), but no modified conjugation law is given and no calculation shows that the two charges in (4.16) are conjugate or that (4.18) is Hermitian. The manuscript also does not display the super-Poincare bracket of the new charges or their conformal partners. A corrected resubmission must define every symbol, prove charge conjugacy and Hamiltonian Hermiticity, and show the complete $OSp(\mathcal{N}|2)$ closure rather than state that the coefficients were fixed by insisting on it.

7. **The spin phase space and the physical status of the enlarged models need to be specified.**

   Section 3.2 sets $g = 0$ before the current rewriting. The spin extension is therefore not shown for the general ECM coupling that appears in Sections 2 and 3.1, despite the broader language of the abstract and conclusion. The authors should either construct the nonzero-$g$ extension or delimit the claim explicitly.

   Equations (3.8) and (4.15) introduce $v, \bar{v}, w, \bar{w}$ as commuting variables with opposite symplectic signs, but the manuscript does not state their reality conditions, phase-space constraints, gauge identifications, or allowed values of the bosonic Casimirs. It is consequently unclear whether the construction describes an unconstrained noncompact oscillator phase space, a coadjoint orbit, or a semi-dynamical spin sector after a number constraint. This information is needed both for Hermiticity and for the quantum Hilbert space.

   The paper begins from integrable ECM and Calogero systems, yet it does not ask whether the new bosonic-current couplings preserve a Lax representation or a complete set of commuting charges. Nor does it discuss ordering corrections when the classical current products and inverse-square Hamiltonians are quantized. The authors should define the classical spin phase space, identify the relevant Casimir leaves, discuss integrability, and state the scope of the construction at the quantum level. Without this, the physical significance of adding the extra bosonic variables remains difficult to assess.

8. **The novelty claims require a sharper comparison with the cited and recent literature.**

   The two-body specialization itself is not the main novelty, and the $OSp(\mathcal{N}|2)$ symmetry of the authors' earlier supersymmetric Calogero family was already established in the work cited as reference [24]. General $SU(1,1|n)$ mechanics and current-valued extensions are also discussed in the works cited as references [14], [19], and [20]. Equation (3.18) is explicitly said to coincide with Kozyrev's model. The manuscript must therefore identify whether the unreduced equations (3.10)–(3.13), once corrected, are canonically inequivalent to that model, whether they allow more general bosonic $U(\mathcal{N})$ representations, and what observable or structural advantage the additional fermions provide.

   The paper should also compare with Kozyrev and Krivonos, *New variants of $N = 3, 4$ superconformal mechanics*, arXiv:2510.00916, which is not cited and explicitly uses additional fermions to obtain maximal superconformal symmetries. That work makes the broad conclusion in Section 5 less novel than the manuscript suggests. The real potential advance here is the ECM-derived hidden enhancement for arbitrary even $\mathcal{N}$ and, if they can be repaired, specific parent embeddings and spin extensions. The introduction and conclusion should be reorganized around those precise results.

   Finally, the assertions that the Calogero construction has "purely" $OSp(\mathcal{N}|2)$ symmetry and that one "has to" use extra fermions are stronger than what is proved. The manuscript constructs candidate realizations; it provides neither a no-enhancement theorem for the Calogero case nor a necessity theorem excluding other phase spaces. These claims should either be proved or replaced by accurately delimited constructive statements.

## Minor and presentation comments

1. In equation (2.14), the relative fermion is written as $\psi_a = \psi_1^a - \psi_2^a$; its free index should be consistent with the upper index used everywhere else.

2. Equation (3.2) says that its Hamiltonian is equation (2.7), but equation (2.7) contains the parent supercharges. The intended reference appears to be equation (2.20).

3. Section 3.3 should define separate ranges and roles for Latin $a, b$ and Greek $\alpha, \beta$ indices. In equation (3.24), the definition of $\mathfrak{q}^\alpha$ uses $Q^a$ and $q^a$ on the right-hand side, apparently where an $\alpha$ index is intended.

4. Since every complex supersymmetry index runs from 1 to $\mathcal{N}/2$, the construction assumes even $\mathcal{N}$. This restriction should be stated in the abstract, main claims, and conclusion.

5. The objects in equation (4.2) obey a $u(2)$ matrix-current algebra unless a traceless condition is imposed; the trace is central. Calling it $su(2)$ requires stating the constraint or projecting out the trace.

6. The fermionic bracket immediately before equation (4.1) repeats the index $a$ on the left while displaying $\delta^a_b$ on the right. The barred fermion should carry the independent index $b$.

7. Equation (4.12) uses $(J_f)^a{}_b$, although the defined generator in equation (4.10) is $(V_f)^a{}_b$.

8. Equation (4.11) should display the Kronecker delta multiplying $D$, and its final barred charge has the wrong index position.

9. In Appendix equation (A.1), the last sign in $\{T^{ab}, \bar{T}_{cd}\}$ disagrees with equation (2.24). The final term should be checked together with the corrected bosonic-current conventions.

10. The manuscript needs a systematic notation and language pass. Examples include "superymmetry," "two-particles case," "holographuc," "papaer," "deaply," and "cunclude." Several bibliography entries also contain damaged dash characters or malformed arXiv brackets. These issues are secondary to the algebraic corrections but are too frequent for a journal version.