Referee Report

Multi-rotating black holes with non-aligned angular momenta in 5D Kaluza–Klein theory

arXiv:2607.11609

Recommendation: Reject in the present form; a fundamentally revised analysis could be considered as a new submission. The harmonic-function construction is economical, and a genuinely regular stationary family with independently oriented angular momenta would be an interesting addition to the literature. The manuscript does establish a plausible local solution of the field equations away from the point sources, and its exterior positivity estimates contain useful ingredients. However, the central interpretation of the point sources as smooth extremal black-hole horizons is not established and appears to fail for the generic non-aligned configurations claimed in the title. The coordinate change used in Eqs. (59)–(62) contains a logarithmic azimuthal shift. The non-axisymmetric multipoles generated by the other centers then produce log-oscillatory metric coefficients; in the displayed chart the radial component does not even have a generic limit at the putative horizon, while an angular component is generically only once differentiable. The leading scaling metric in Eq. (63) discards precisely these terms and therefore cannot prove horizon regularity. In addition, the five-dimensional uplift is not globally defined without circle-bundle patching and charge quantization, no five-dimensional horizon analysis is supplied, and the physical identification of each vector parameter as an individual angular momentum is not justified in a spacetime without rotational Killing fields. These are not matters of presentation: they affect whether the proposed objects are regular black holes at all.

Summary and overall assessment

The paper begins by reviewing the aligned multi-center Kaluza–Klein solutions and the reduction of five-dimensional vacuum gravity with the commuting Killing vectors ∂_t and ∂_{x^5} to the Maison $SL(3,\mathbb{R})$ sigma model. For a flat three-dimensional base, the coset representative is written in Eq. (36) in terms of two harmonic functions f and g and a fixed matrix A. The new step is to replace the aligned dipole harmonic by

$$f=\sum_i \frac{M_i}{|\boldsymbol{x}-\boldsymbol{x}_i|}, \qquad g=\sum_i \frac{M_i^2\boldsymbol{J}_i\cdot(\boldsymbol{x}-\boldsymbol{x}_i)}{2P_iQ_i|\boldsymbol{x}-\boldsymbol{x}_i|^3},$$

with arbitrary constant vectors $\boldsymbol{J}_i$. Equations (52)–(57) then give the rotation one-form, the Kaluza–Klein monopole connection, the five-dimensional metric, and the reduced four-dimensional Einstein–Maxwell–dilaton fields.

Section V argues that every puncture is an extremal horizon, reads off total asymptotic charges, proves $H_\pm>0$, and seeks to exclude closed timelike curves by proving positivity of the spatial block of the four-dimensional metric. The paper finally specializes to two centers and notes that generic spin directions destroy axial symmetry. The algebraic observation behind Eqs. (50)–(53) is promising: away from the sources, a dipole of any spatial orientation is harmonic, and the fixed nilpotent sigma-model generator makes such superposition possible. The exterior metric is also manifestly smooth in Cartesian coordinates wherever $H_\pm>0$. Thus the manuscript likely contains a valid family of local stationary solutions on the punctured exterior domain.

That is not yet a black-hole solution. The essential issue is the extension across every puncture. For extremal rotating multi-center geometries, agreement of the leading near-horizon limit with a known one-center solution is insufficient: subleading multipoles can determine the differentiability of the full horizon. Here those multipoles are generically non-axisymmetric in the rotating frame required at each center. The manuscript neither constructs Gaussian-null coordinates to adequate order nor tests a curvature or parallelly propagated invariant. In fact, its own proposed coordinates expose a direct obstruction, detailed below. The result is especially serious for the explicit two-center configuration: when a spin is not parallel to the separation axis, the first external mass multipole already has the offending azimuthal dependence.

If the authors can identify a nontrivial subclass with sufficiently smooth horizons, define its five-dimensional bundle globally, and establish the physical charges, such a restricted result could be worthwhile. The present claim of arbitrary non-aligned angular momenta and complete regularity is not supported.

Major comments

1. **Equations (59)–(63) do not establish a smooth horizon; for generic data the displayed "regular" chart is not even continuous.**

   Consider the ith center, translate it to the origin, and align the local z-axis with $\boldsymbol{J}_i$, as in the manuscript. Let $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and let $\boldsymbol{d}_{ij} = \boldsymbol{x}_j - \boldsymbol{x}_i$. Equation (50) has the local expansion

   $$f = \frac{M_i}{r} + f_0 + r, u_1(\theta,\phi) + O(r^2), \qquad u_1 = \sum_{j\neq i} \frac{M_j\, \boldsymbol{d}_{ij}\cdot\boldsymbol{n}}{|\boldsymbol{d}_{ij}|^3}.$$

   Unless every displacement $\boldsymbol{d}_{ij}$ is parallel to $\boldsymbol{J}_i$, u_1 contains $m = \pm 1$ terms such as $\sin\theta\cos(\phi - \phi_{ij})$. Write

   $$S := \sqrt{H_+ H_-}, \qquad \Delta_i(\theta) := \sqrt{P_i^2 Q_i^2 - J_i^2\cos^2\theta}.$$

   Using the product displayed in Eq. (78), one obtains

   $$S = \frac{2\Delta_i}{r^2} + \frac{S_{-1}(\theta)}{r} + S_0(\theta,\phi) + O(r),$$

   where the contribution of the external mass dipole is

   $$\delta S_0 = \frac{4P_i^2 Q_i^2}{M_i \Delta_i(\theta)}, u_1(\theta,\phi).$$

   This coefficient follows directly from the $\sin^6(2\alpha) f^4$ term in Eq. (78); before taking the square root, its r^{-2} non-axisymmetric contribution is $4M_i^3 \sin^6(2\alpha) u_1$.

   Equation (59), together with Eq. (62), integrates to

   $$\phi = \phi' + b_0 \log r + \text{constant}, \qquad b_0 = \frac{2J_i}{a_0}.$$

   For a rotating center $b_0 \neq 0$. The supposedly regular metric component in the θ direction therefore contains

   $$g_{\theta\theta} = Sr^2 - S^{-1}(\omega_\theta^0)^2 = 2\Delta_i + rS_{-1} + r^2 \delta S_0(\theta, \phi' + b_0\log r) + O(r^2).$$

   The external part of ω_θ^0 is $O(r)$, so its displayed contribution begins only at $O(r^4)$ and cannot cancel this term. A function $r^2\cos(\phi' + b_0\log r)$ is C^1 but not C^2 at $r = 0$: its second radial derivative oscillates without a limit. Thus the chart proposed by the paper already fails the minimum differentiability needed to discuss a classical curvature tensor.

The same phase is not confined to a convenient metric component. With $q_i = J_i/(P_i Q_i)$, Eq. (57) gives, in the same natural ingoing transverse parameter, a non-axisymmetric scalar contribution of the form

$$\delta\phi = -\frac{2\sqrt{3}\, q_i \cos\theta}{M_i(1 - q_i^2\cos^2\theta)} r^2 u_1(\theta, \phi' + b_0 \log r) + \cdots .$$

The norm of ∂_{x^5} in the uplift carries equivalent information. This provides an invariant warning that a change of individual metric components cannot be accepted as a smoothness proof.

There is an even more immediate problem. Set $K = 1 + b_0^2 \sin^2\theta$. The choices in Eqs. (60)–(62) cancel the r^{-2} and r^{-1} pieces of g_{rr}. Varying the remaining finite term with respect to the non-axisymmetric δS_0 gives two equal contributions, one from SK and one from the inverse factor in $-S^{-1}(dt + \omega^0)^2$. Consequently the chart contains generically

$$\delta g_{rr} = 2K\, \delta S_0(\theta, \phi' + b_0 \log r) + o(1),$$

which has no limit as $r \to 0$. The coefficient a_1 in Eq. (61) cancels an axisymmetric r^{-1} divergence; it does not address this external multipole.

Equation (63) remains a formal leading near-horizon scaling limit because all of these subleading terms scale away. That is exactly why the limit cannot establish smoothness of the original metric. A more elaborate, angle-dependent coordinate transformation might rearrange individual components, but the Fourier modes acquire factors r^{imb_0}, and the r^2 mode above gives a generic finite-differentiability obstruction. Closely related behavior is known in rotating multi-center geometries; see G. N. Candlish, “On the smoothness of the multi-BMPV black hole spacetime,” arXiv:0904.3885.

The authors must construct full Gaussian-null coordinates, including all non-axisymmetric subleading terms, and prove at least a C^2 extension (preferably the smooth or analytic extension claimed), or use a coordinate-invariant quantity along affinely parametrized geodesics to determine the actual differentiability. If this cannot be done, the family must be restricted to configurations in which all relevant external multipoles are axisymmetric about each local spin axis. For two genuinely nonparallel spins, that restriction is generically impossible. Until this issue is resolved, the statements after Eq. (63), in the abstract, and in Section VI that the punctures are regular black-hole horizons are unjustified.

2. **The five-dimensional metric is not globally specified, and the four-dimensional analysis does not prove regularity of the Kaluza–Klein uplift.**

   Near one center, the monopole potential in Eq. (53) is

   $$\widetilde{\omega}_i^5 = -2P_i \cos\theta_i \, \mathrm{d}\phi_i.$$

   This is not a global one-form on the surrounding two-sphere. North and south patches differ by $4P_i \mathrm{d}\phi_i$, which must be absorbed by a transition of the compact coordinate x^5. Since the paper assigns x^5 period $2\pi R_{KK}$, consistency of the transition around $\phi_i \mapsto \phi_i + 2\pi$ requires

   $$\frac{4P_i}{R_{KK}} \in \mathbb{Z}$$

   in the conventions of Eqs. (1) and (53). The resulting integer also determines the circle-bundle Chern class and the lens-space topology of the five-dimensional horizon cross-section. None of this is discussed. With $P_i = 2M_i \cos^3\alpha$, arbitrary continuous masses do not generically satisfy these conditions for one fixed compactification radius.

Section V.A studies only the reduced four-dimensional metric. It does not show that the full metric (54) is smooth across patch boundaries or at a horizon, determine the five-dimensional horizon topology, exclude orbifold defects, or distinguish asymptotically flat five-dimensional geometry from asymptotically Kaluza–Klein geometry with a nontrivial circle bundle. The title and repeated claims about five-dimensional black holes require this global analysis. Alternatively, the scope and title should be changed so that the primary result is explicitly a local or four-dimensional Einstein–Maxwell–dilaton solution, with the conditions for a global uplift stated separately. The statement in Section V.E that only one Killing vector remains is also true only after reduction: the five-dimensional metric always retains ∂_{x^5} in addition to ∂_t.

3. **The parameters $\boldsymbol{J}_i$ have not been shown to be individual physical angular momenta.**

   Equations (64)–(69) determine only the total ADM mass, total ADM angular-momentum vector, and total charges at infinity. In the generic solution there is no rotational Killing vector, as the authors emphasize in Section V.E. A Komar angular momentum associated with each center is therefore not automatically available, and decomposing the total ADM vector into contributions from different holes can be ambiguous because the gravitational and electromagnetic fields also carry angular momentum.

   The residue in Eq. (51) and the dipole coefficient in Eq. (52) certainly define parameters $\boldsymbol{J}_i$. To call each one the spin of a black hole, the authors should construct a smooth horizon cross-section and evaluate an appropriate quasi-local angular momentum, specifying the local axial vector and its normalization, or provide an equivalent invariant charge construction. The same section should verify the individual electric and magnetic fluxes and explain the status of M_i as a horizon mass. Without this, the condition $|\boldsymbol{J}_i| < |P_i Q_i|$ is a bound on source parameters, not yet a demonstrated physical slow-rotation bound for each constituent.

   This issue also affects the claimed equilibrium interpretation. Section V.E admits after Eq. (88) that the absence of conical singularities in the non-aligned case has not been demonstrated, and Section VI leaves the force balance to future work. In the reduced metric (55), ordinary points away from the punctures are manifestly smooth in global Cartesian coordinates once $H_\pm > 0$, so a Weyl-axis test is neither necessary nor the right generic diagnostic. The actual global problems are the horizon extension and the Kaluza–Klein patching. The authors should state precisely what distributional or holonomy defect they mean by a "conical singularity" without an axial Killing field and prove that the field equations hold without such defects. It is inconsistent to market a completely regular balanced configuration while explicitly leaving that question unresolved.

4. **The two-harmonic sigma-model ansatz needs an additional algebraic condition and a direct verification for the chosen matrix.**

   For Eq. (36), the two exponentials commute and

   $$\chi^{-1}\partial_i\chi = A\,\partial_i f + A^2\,\partial_i g.$$

   Harmonicity of f and g then gives Eq. (34), but the flat-base constraint (35) becomes

   $$\mathrm{tr}(A^2)\,\partial_i f\partial_j f + \mathrm{tr}(A^3)(\partial_i f\partial_j g + \partial_i g\partial_j f) + \mathrm{tr}(A^4)\,\partial_i g\partial_j g = 0.$$

   The three conditions printed in Eq. (38) set only the first coefficient to zero; they do not by themselves imply $\mathrm{tr}(A^3) = \mathrm{tr}(A^4) = 0$. The specific matrix in Eq. (39) appears to be nilpotent with $A^3 = 0$, which would repair the argument, but this property is neither stated nor proved. The authors should correct the general statement following Eq. (38), show explicitly that Eq. (39) has the required nilpotency, and then verify Eqs. (34)–(35). They should also state that the vacuum equations are being solved on the punctured base and explain how the distributional harmonic sources are replaced by regular horizons; the latter step is precisely where the smoothness problem enters.

5. **The regularity and causality arguments require sharper hypotheses and do not cure the horizon problem.**

   The construction contains $1/\sin\alpha$ and $1/(P_iQ_i)$, while the positivity proof assumes $M_i > 0$. A theorem-like statement of the allowed parameter domain is needed: the centers should be distinct, $M_i > 0$, $P_iQ_i \neq 0$, the range of α should be fixed (normally $0 < \alpha < \pi/2$ for the charge assignment used), and the strict spin bound and Kaluza–Klein quantization conditions should be listed. The endpoint limits with vanishing electric or magnetic charge cannot simply be included in the displayed formulas. All centers also share one value of α, and hence one charge-to-mass ratio and common charge signs; the scope should not suggest independently variable dyonic charges. In Eq. (72), the first displayed strict inequality becomes an equality at $\alpha = \pi/4$, although the later bound still yields strict positivity. The chain should be written with the correct weak or strict signs.

   Equation (71) does not prove that curvature singularities occur "precisely" at zeros of H_+H_-. A proportionality sign displaying only a denominator omits the numerator and its possible zeros, and component divergence is not a coordinate-invariant singularity test. Away from the punctures, a cleaner argument is available: f, g, and ω^0 are analytic, so positivity of $H_\pm$ makes the Cartesian four-dimensional metric analytic. At a puncture, however, only a genuine extension or an invariant curvature analysis can establish regularity; positivity of $H_\pm$ is not enough.

   The lower bound from Eq. (78) to Eq. (79) also suppresses several terms without explaining why they are positive. The omission can likely be repaired. From the spin bound one has

   $$|g| \leq \sum_i |g_i| < \frac{1}{2}\sum_i f_i^2 \leq \frac{1}{2} f^2,$$

   so $6f^2 + 4\cos(2\alpha)g > 4f^2$, and the remaining discarded mixed monomials from f^3 and f^4 are positive for $f_i > 0$. These steps should be written explicitly. Once Eq. (77) is proved in the exterior, the strongest and clearest conclusion is that dt has timelike gradient there, yielding stable causality. The phrase "on the horizons" must be withheld until a sufficiently differentiable horizon extension has been constructed. The corresponding five-dimensional statement should then be proved separately on the globally patched circle bundle.

6. **The novelty claim must be reassessed after the differentiability and global-regularity questions are settled.**

   The replacement of an aligned dipole by the vector dipoles in Eq. (51) is a concise and potentially useful observation, and the resulting punctured-domain solution appears to go beyond the axisymmetric families cited in the manuscript. The main physical novelty, however, is not the formal harmonic superposition; it is the assertion that the generic punctures remain regular black holes after rotational symmetry is broken. That is exactly the assertion for which no proof is given and for which the logarithmic analysis above supplies contrary evidence.

   The discussion should engage with the established smoothness literature for extremal multi-center horizons, including the mechanism analyzed in arXiv:0904.3885 and the static analysis of G. N. Candlish and H. S. Reall, arXiv:0707.4420. These results are not identical theories, but they make clear why a leading near-horizon geometry is not a smoothness proof and why broken rotational symmetry can lower differentiability. If the generic family has only a C^1 or weaker null boundary, the title, abstract, and publication claim must change accordingly; it would not meet the usual standard for a classical solution of the Einstein equations with a regular black-hole horizon.

Minor comments

1. Equation (61) should use the relative distances $|\boldsymbol{x}_j - \boldsymbol{x}_i|$, after explicitly translating the ith center to the origin. As written, $|\boldsymbol{x}_j|$ depends on an unstated coordinate choice. The derivation should also show all orders that remain after the cancellations, rather than merely list a_0, a_1, and b_0.

2. The labels for the four-dimensional metric, gauge field, and dilaton are duplicated in Eqs. (7)–(9) and (55)–(57). The compiled manuscript reports multiply defined labels. All labels should be unique and all cross-references checked.

3. The sentence immediately after Eq. (51) cites Ref. [5] as the previous aligned Kaluza–Klein solution. Ref. [5] is the authors' different work on asymptotically locally Euclidean Myers–Perry configurations; the intended citation appears to be Ref. [16].

4. The manuscript should distinguish consistently between four-dimensional asymptotic flatness and five-dimensional Kaluza–Klein asymptotics. With compact x^5 and magnetic charge, the five-dimensional asymptotic circle is globally twisted and the spacetime is not ordinary five-dimensional Minkowski space.

5. The harmonic function f, the near-horizon function $f(\theta)$ below Eq. (63), and several unrelated uses of g make the central calculations unnecessarily hard to follow. Distinct symbols would help, especially in the horizon expansion.

6. The normalization used to call P_i and Q_i physical charges should be stated through explicit surface integrals. This would also make the factors of two in Eq. (65) and the bundle condition transparent.

7. In the aligned two-center discussion, the sentence following Eq. (87) should read $\omega^0 = \omega_\phi \, \mathrm{d}\phi$. More generally, the paper needs a careful language edit for subject-verb agreement, missing articles, and several malformed sentences in Sections I and V.E.

8. The paper should state whether saturation $|\boldsymbol{J}_i| = |P_i Q_i|$ is excluded only because the area vanishes or because the local geometry develops a null or curvature singularity. The strict inequality is used repeatedly, and its geometric limiting behavior should be described.

Conclusion

The exterior harmonic construction is interesting, but the manuscript has not shown that its generic centers are regular black holes. The log-oscillatory multipoles generated by Eqs. (50), (59), and (62) invalidate the displayed horizon chart, while the five-dimensional uplift lacks the required global bundle data. These problems strike at correctness and at the claimed novelty. A publishable revision would need a full differentiability analysis of every horizon, a global construction of the compact fifth direction, invariant definitions of the individual charges and spins, and a correspondingly narrower statement of the parameter space and physical conclusions.