Referee Report

on “Bethe Ansatz without Nesting”

Summary of the paper

The manuscript proposes an auxiliary-root-free description of rational $\mathfrak{gl}_\ell$ spin chains with vector representations at the sites. The basic idea is to retain only the first Baxter polynomial $Q_1(u)$, whose zeros are the momentum-carrying or principal Bethe roots, and to eliminate the higher Baxter polynomials that appear in the conventional nested Bethe ansatz. The authors combine the quantum spectral curve, the fusion relations of the fundamental transfer matrices, and Lagrange interpolation at the inhomogeneities.

After reviewing the nested equations and the quantum spectral curve in Section 2, the manuscript treats $\mathfrak{gl}_3$ explicitly. Evaluating the third-order quantum spectral curve at a zero of Q_1, and reconstructing T_1 from its values at the inhomogeneities, gives equation (3.13). A residue resummation then produces the homogeneous-limit-compatible equation (3.20). Formulas (3.24) and (3.27) reconstruct T_1 and T_2 from the principal roots. Section 4 carries out an analogous calculation for $\mathfrak{gl}_4$, leading to equation (4.9), formulas for the numbers of auxiliary roots, and a proposed reconstruction of T_3.

The main all-rank proposal is equation (5.2). It is expressed through reduced twist characters and iterated Cauchy-kernel sums built from the coefficients $\mathcal{C}_n$. The authors establish a rank-embedding property, write untwisted and dual forms, and introduce a recursive hierarchy Φ_ν in equation (5.32). A proposed transfer-matrix representation, equation (5.33), together with the highest closure (5.40), is then used to recover (5.2) from cancellation of apparent poles. Section 6 studies the quasi-classical limit for $\mathfrak{gl}_3$ and rewrites the resulting equation (6.14) as a no-pole condition for a candidate third-order scalar operator. Finally, Appendix A compares nested and non-nested roots in one $N = 3$, $\mathfrak{gl}_3$ example and lists 27 solutions.

General assessment and recommendation

The central idea is attractive. A formulation involving only Q_1 would give a conceptually economical view of higher-rank spectra, and the explicit $\mathfrak{gl}_3$ and $\mathfrak{gl}_4$ calculations contain useful material. In particular, the passage from the root evaluation of the quantum spectral curve to the regular form (3.20), and the attempt to organize the reconstructed transfer matrices into a uniform hierarchy, should be of interest to the integrability and mathematical-physics communities.

However, the strongest claims of the paper are not yet established. The arbitrary-rank equation is inferred from low rank, while the transfer-matrix formula used to justify it is explicitly conjectured. The manuscript also shows necessity much more clearly than sufficiency: it does not prove that every solution of the proposed rational equations gives a genuine and unique Bethe-algebra eigenvalue. Several displayed equations in the central derivations are internally inconsistent, and the numerical appendix prints the nested equations with the wrong signs. The Gaudin expansion also omits terms at intermediate steps, even though the omitted terms appear to cancel from the final equation.

I therefore recommend **major revision**. I would be willing to reconsider the manuscript after either a non-circular proof of the all-rank construction or a systematic reframing of that construction as a conjecture, together with a proof or precise theorem addressing completeness, correction of the formulas listed below, and reproducible numerical evidence. If these issues are resolved, the work could become suitable for publication.

Major comments

1. **The arbitrary-rank result is presently conjectural, and the regularity argument does not remove that gap.**

   Equation (5.2) is introduced as the natural generalization of the $\mathfrak{gl}_3$ and $\mathfrak{gl}_4$ results. More importantly, equation (5.33) is explicitly described as a conjectured representation of T_ν, and the final-node relation (5.40) is motivated by what one "may expect." Equations (5.47)–(5.51) then show that, *assuming* (5.33) and (5.40), polynomiality of the highest transfer matrix is equivalent to $\mathcal{R}_\ell = 0$. This is a valuable consistency check, but it is not an independent derivation of either assumption. The embedding result only proves that a rank-$(\ell-1)$ solution is also a rank-ℓ solution; it does not prove that (5.2) is the correct or complete rank-ℓ spectral equation.

   The principal claim in the abstract and conclusions should therefore be changed unless the missing induction is supplied. A satisfactory proof should derive (5.33) from the established fusion and interpolation data, rather than infer it from the first few ranks. In particular, it should show for every ν that the proposed expression has degree νN, the correct kinematic zeros and leading character, all required fusion values, and hence coincides uniquely with the physical transfer-matrix eigenvalue. The highest closure (5.40) should then be derived independently from the quantum determinant or the full quantum spectral curve. If this cannot be done, the paper should consistently present (5.2), (5.33), and (5.40) as conjectures supported by low-rank calculations, including in the title-level claims, abstract, Introduction, and Conclusions.

2. **A necessary root equation is not yet a complete spectral characterization.**

   For a genuine eigenstate, the quantum spectral curve implies the low-rank root equations. The converse is not demonstrated. In particular, the paper does not prove that an arbitrary solution of (3.20), (4.9), or (5.2) reconstructs polynomial transfer matrices satisfying the full functional system, belongs to the Bethe algebra, and produces a nonzero eigenvector through the B-operator. Evaluating a functional identity only at the zeros of Q_1 gives necessary conditions; an argument is needed that no additional conditions are lost and no extraneous branches are introduced.

   The footnote below equation (3.13) defines admissibility only by excluding coincident roots. This is not enough for the displayed rational equations. One must also control roots separated by η, zeros of $Q_1(v_k \pm \eta)$, roots on the inhomogeneities or their shifts, zeros of $Q_1(u_j)$, and roots at infinity. The allowed range of m_1 and all genericity assumptions should be stated. The authors should either construct the higher polynomial Q-functions from every admissible solution or invoke and verify the hypotheses of a precise separation-of-variables completeness theorem. The empirical integrality of (3.39) and (4.24) is not a replacement for existence of polynomial auxiliary Q-functions.

   This point is especially important in the homogeneous and untwisted limits. A smooth limiting formula does not by itself establish completeness when twist eigenvalues collide, transfer-matrix eigenvalues become degenerate, descendants can be represented by roots at infinity, and finite roots can collide. In the untwisted theory, Q_1 may label a multiplet eigenvalue rather than every individual state in that multiplet. The claim that the "complete spectral data of an eigenstate" are encoded by Q_1 should be restricted to the precise generic setting that is proved, with a separate statement about limiting multiplets.

3. **Several central displayed formulas must be corrected before the derivations can be checked reliably.**

   I list the most consequential inconsistencies here.

   - The unlabeled equation (3.4) should read $\mathsf{T}_2(u_j + \eta)$ on its left-hand side, not $\mathsf{T}_2(u_j)$. The displayed right-hand side follows by evaluating T_2 at $u_j + \eta$, and it is this shifted value that is needed in the second fusion relation (3.5).
   - Equation (4.4) omits the spectral shifts in the quantum determinant. The formula consistent with (2.9) is
   $$\mathsf{T}_4(u) = \chi_4\, P_1(u)P_2(u+\eta)P_3(u+2\eta)P_4(u+3\eta).$$
   Equation (4.5) appears to use this shifted expression implicitly, so the printed equations are mutually inconsistent.
   - Equations (3.37) and (3.38) give the constant term of $\mathcal{A}(u)$ as $\delta_1 + \delta_3$. From (3.25) and from the preceding nested expression it is $\bar{\chi}_1 = \delta_2 + \delta_3$. Equation (3.39) implicitly uses the latter value.
   - In equation (4.12), substitution of (4.11) into the inner sum requires $(\delta_2 + \delta_3)/\delta_1$, whereas the first line prints $\delta_2\delta_3/\delta_1$. The next line effectively reverts to the sum, so the two lines do not follow from one another as written.
   - Equation (4.17) is ill-defined: the summation condition $n \neq l$ leaves l unbound, while the product over all l includes the singular factor $l = n$. The intended expression appears to be a sum over n with a product over $l \neq n$, but it should be rederived and printed unambiguously.
   - Equations (5.13) and (5.14) define functions of the external variable v_k but use $v_n - v_{n_1}$ in the first Cauchy denominator, with n free. That denominator should be $v_k - v_{n_1}$.
   - Equation (5.36) is shift-inconsistent. If $\Phi_1(u) = \mathcal{A}(u-\eta)$, then the denominator in its sum is $u - v_n$, not $u - v_n + \eta$. The latter gives $\mathcal{A}(u)$.
   - The highest fusion relation stated after (5.40) is missing a shift. It should be $\mathsf{T}_1(u_j)\mathsf{T}_{\ell-1}(u_j + \eta) = \mathsf{T}_\ell(u_j)$, in accordance with (3.6).
   - Equation (3.21) contains the free index l in $\prod_j (z - u_l - \eta)$; the factor should depend on u_j. The notation for $\mathcal{C}_n$ also changes in equations (4.10)–(4.16), and the twist symbols δ_1 and alternate delta glyphs are interchanged repeatedly. These may be typographical errors, but they occur inside the principal formulas and should be corrected systematically.

   The revised manuscript should include an equation-by-equation check of the low-rank construction. At present some later statements use the apparently intended formula rather than the formula actually printed, making it difficult to determine which expressions were tested.

4. **The appendix equations have the wrong signs, and the numerical evidence is not documented at the level needed for a completeness claim.**

Both right-hand sides in equation (A.1) have an erroneous minus sign. At a root $u_k^{(a)}$,

$$\frac{Q_a(u_k^{(a)}-\eta)}{Q_a(u_k^{(a)}+\eta)} = -\prod_{n\neq k}\frac{u_k^{(a)}-u_n^{(a)}-\eta}{u_k^{(a)}-u_n^{(a)}+\eta}.$$

This self-root minus sign cancels the explicit minus sign on the right of equation (2.4). Consequently, the two right-hand sides of (A.1) should be $+\delta_1/\delta_2$ and $+\delta_2/\delta_3$. The tabulated roots themselves appear to use the plus convention. For example, for the listed $(m_1, m_2) = (1, 0)$ root $v = 2.28641 - 6.31564\,i$, the parameters of Appendix A give

$$\frac{\delta_1}{\delta_2}\frac{P_1(v)}{P_2(v)} \simeq 0.9999999 + 2.2\times 10^{-7} i,$$

not -1. Thus the table is not a numerical solution of the nested equations as they are printed.

Because Appendix A is the main evidence for equivalence and completeness, the numerical procedure must also be described. The revised paper should state the solver, working precision, residual tolerance, treatment of root permutations and denominator loci, and how the authors certified that all solutions were found. Agreement of rounded decimal entries is not "exact" agreement. Direct diagonalization of short chains and comparison of the reconstructed transfer matrices would provide a more independent test.

The assertion in Section 5 that $\mathfrak{gl}_5$ tests showed complete agreement in all cases considered is not reproducible without the chain lengths, twist and inhomogeneity parameters, sectors, solution counts, and tolerances. The paper should provide explicit data for several generic examples at ranks three, four, and five. Such tests are supporting evidence, not a substitute for the proof requested in comments 1 and 2.

5. **The Gaudin expansion and the claimed oper identification need revision.**

For the unshifted quantities used in Section 6, equation (6.4) is missing a derivative term. The expansion is

$$\mathcal{C}_k = 1 + \eta(V_k - P_k) + \eta^2\left[\frac{1}{2}(V_k - P_k)^2 + \frac{1}{2}P_k' - \frac{1}{2}W_k\right] + O(\eta^3).$$

Accordingly, S_1 contains an additional $\frac{1}{2}\eta^2 P_k'$ beyond the B_k printed in (6.6). The driving expansion (6.7) simultaneously omits the compensating contribution; its second-order bracket should contain $-\frac{1}{2}P_k'$. These two omitted terms cancel when the two sides are combined, so equations (6.9) and (6.14) may survive, but the displayed derivation is not algebraically correct and must be redone. In addition, the subset sum in (6.1) should involve $(\tau_{\alpha_1} + \cdots + \tau_{\alpha_\nu})^r$, rather than the consecutive sum ending at α_ν, and the remainders in (6.2) should be $O(\eta^3)$ because second-order terms are already displayed.

The oper interpretation is currently suggestive rather than established. Once a_1, a_2, and Q_1 are specified, one can always define

$$a_3(u) = -\frac{Q_1'''(u) + a_1(u)Q_1''(u) + a_2(u)Q_1'(u)}{Q_1(u)}.$$

Equation (6.14) is then precisely the condition that this rational function have no pole at a zero of Q_1. To show that the resulting operator is the relevant Gaudin oper, the authors should display a_3, establish its allowed singularity structure and local exponents at the inhomogeneities, analyze the point at infinity, and identify the accessory parameters with the Gaudin Hamiltonian eigenvalues. An equation-level comparison with the cited work of Mukhin, Schechtman, Tarasov, and Varchenko is needed to say whether (6.14) is equivalent to their non-nested $\mathfrak{sl}_3$ equations, a change of gauge or variables, or a genuinely new formulation. Until then, statements that the equations "precisely" define the scalar oper should be softened.

6. **The novelty and practical value should be positioned more sharply relative to the cited literature.**

   The single-B state construction, the SoV bases, and the quantum spectral-curve description are established ingredients in the cited work of Gromov–Levkovich-Maslyuk–Sizov, Ryan–Volin, and Maillet–Niccoli. The manuscript also acknowledges an earlier non-nested $\mathfrak{gl}_3$ construction by Liashyk and Slavnov, but leaves the precise relation as an open problem. At least in rank three, the authors should compare equation (3.20) directly with the determinant-based equation and determine whether the two systems are algebraically equivalent. This is necessary to isolate what is new in the low-rank result.

   Likewise, the phrase "truncated Q-system" should be backed by a concrete map to established Q- or QQ-system relations, or else used only as an analogy. Foundational claims currently referred mainly to the 2026 monograph should be accompanied by the relevant primary references. The most plausible new contribution is the explicit all-rank root-level hierarchy and its transfer-matrix reconstruction, which makes a proof of comment 1 particularly important.

   Finally, eliminating auxiliary variables does not automatically make the equations computationally simpler: the iterated sums in (5.2) scale as powers of m_1 as the rank grows. A small comparison of equation count, arithmetic complexity, and numerical performance against the nested system would clarify the intended practical benefit. Even if the main value is conceptual rather than computational, this should be stated accurately.

Minor comments and presentation

1. Consolidate the assumptions at the beginning of the paper: vector representations, nonzero η, generic distinct inhomogeneities, a simple twist spectrum, simple Bethe roots, and all additional nonvanishing conditions needed by the rational formulas.

2. The statement that the dual equation (5.5) follows by applying the inverse Cauchy kernel should include the inverse formula and its domain of validity. Invertibility can fail on exceptional root configurations, which is also relevant to admissibility.

3. Equation (2.11) has inconsistent quantum-minor indices: the summand mixes k with ν and does not use the displayed upper β indices consistently. This definition should be repaired even if it is not used in the later calculations.

4. The statement surrounding equation (2.12) should specify the twist frame and genericity assumptions under which the B-operator construction is known to produce all nonzero on-shell states.

5. In the discussion of the contour function below equation (3.16), $v_k + \eta$ is the pole associated

with the fixed external index k, not a simultaneous family of such poles. The free-index error in (3.21) should be corrected at the same time.

6. Below equation (4.18), "$k = 1, \dots, k$" should read "$k = 1, \dots, m_1$." The manuscript should also use one notation for m_1, η, the twist eigenvalues, and $\mathcal{C}_n$ throughout.

7. Correct the numerous textual errors, including "principle roots," "pricipal," "inhomogeneitis," "conclud," "obatin," "ransfer matrices," the duplicated "the the" and "and and," and the corrupted characters in "Yang–Yang" and "Bäcklund."

8. The comparison table should report enough digits to support its stated tolerance, or provide machine-readable data separately. It would also be helpful to list the reconstructed m_2 or m_3 values next to each non-nested solution rather than asking the reader to infer the matching.

Recommendation

Major revision. The explicit low-rank calculations and the proposed hierarchy are promising, but publication should await either a rigorous, non-circular arbitrary-rank derivation or a substantial reframing as a conjectural construction. The authors should also establish the precise scope of spectral completeness, correct the central equations and Appendix A, repair the Gaudin expansion, and substantiate the oper and novelty claims.