Referee Report

"Holographic Timelike Entanglement and Subregion Complexity in Localized $\mathrm{AdS}_3 \times S^3 \times T^4$ Black Holes"

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Summary and recommendation

This manuscript studies timelike entanglement entropy and a proposed timelike subregion complexity in the localized D1–D5 black-pole geometry. The central idea is to construct spacelike and timelike radial branches in an effective two-dimensional metric evaluated at an angular label θ_0, and then to lift the resulting profile over the physical internal angle θ. The authors analyze a large-radius regime and the exact black-pole functions, emphasize the folded time map $T(r_0, \theta_0)$, and select competing branches only after imposing a common boundary time interval. In the exact geometry they find that the selected branches move toward the cap/horizon transition region, and they interpret the complex lifted area and the real finite volume as complementary probes of localization on the internal sphere.

The topic is interesting and the manuscript contains useful ingredients. In particular, the fixed-boundary-interval bookkeeping in equations (3.100)–(3.107) and (4.69)–(4.74) is more careful than an unrestricted comparison of saddles with different boundary data. The local analysis in Appendix A, including the crossover scale in equation (A.19) and the logarithmic enhancement in equations (A.46)–(A.49), may also be valuable. The figures give a broad view of the competing branch families.

However, I do not think that the main results are presently established. There are two foundational problems. First, the curves obtained at one θ_0 are not shown to extremize the lifted ten-dimensional area. Second, the density used in equations (4.4)–(4.5) is not the induced nine-dimensional volume density of the ten-dimensional metric (2.1). In addition, the exact area formulas have an internal factor-of-two mismatch, the selected branches exhibit serious infrared-cutoff problems, and the numerical analysis does not provide the convergence information needed to separate physical saddle behavior from regulator or grid effects. These issues affect essentially all of the exact complexity results and a substantial portion of the timelike-entanglement conclusions.

My recommendation is therefore **reject in the present form**. I would encourage consideration of a substantially reconstructed manuscript if the authors derive or clearly delimit the lifted variational prescription, correct the volume functional, and repeat the analysis with regulator-independent and reproducible numerics. The required work goes beyond a normal major revision because it can change both the observables and the selected saddles.

Correctness and logical structure

The manuscript proceeds in a logical order: Section 2 defines the black-pole background and the effective metric; Section 3 develops the Lorentzian branch equations, their asymptotic limit, the exact angular lift, and the fixed-T selection; Section 4 repeats this program for the proposed volume observable; and Appendix A analyzes the transition region. The main difficulty is not the organization but the link between these stages. A solution of the reduced problem at a chosen angular label is treated as a candidate saddle of a different, angularly integrated functional without a variational argument. The volume section then reuses the determinant of the effective curve metric as though it were the volume element induced by the original ten-dimensional geometry. Thus the calculations can be internally elaborate while still not compute the claimed observables.

Major comments

1. **The lifted profiles are not shown to be extrema of the ten-dimensional area.**

   Equations (2.22)–(2.23) define a reduced metric at a single label θ_0, and equations (3.14a)–(3.14b) solve the associated one-dimensional problem. Equation (3.60) then holds that profile fixed while integrating the area over every physical angle θ. These are different variational problems. For example, even if one restricts the lifted embedding to be independent of θ, a functional of the schematic form

   $$A[t] = \mathcal{N} \int d\theta \, \sin\theta \cos\theta \int dr \, \sqrt{H(r,\theta) - F(r,\theta) t'(r)^2}$$

   has an Euler–Lagrange equation in which the conjugate momentum is integrated over θ. It is not the reduced geodesic equation evaluated at an arbitrarily chosen θ_0. If the embedding is allowed to have its natural dependence $t = t(r,\theta)$, the full problem is a partial differential equation and includes angular-gradient terms. Minimizing the result over the single parameter θ_0, as in equations (3.64) and (3.107), does not restore stationarity under these functional variations.

   This point is also important for the use of Ref. [6]. In Section 6.1 of that reference, the analogous spatial construction is explicitly described as a restriction to a subset of surfaces, with a changed order of minimization, and hence as an upper-bound estimate rather than a proven RT surface. The present manuscript promotes the construction to an exact timelike-entanglement prescription and extends it to a complex, mixed-signature setting without retaining that limitation or supplying a new derivation. An upper-bound interpretation from the real spatial problem does not automatically survive this extension.

   The authors should begin from the ten-dimensional functional, specify the allowed embeddings and boundary/homology conditions, and derive its Euler–Lagrange and endpoint equations. They should either solve that problem, show that the proposed ansatz is stationary under all relevant variations, or consistently recast the calculation as a controlled restricted-family estimate. Until then, the preference for labels near $\theta_\star$ may be a property of the chosen family rather than of the physical extremal surface.

2. **The timelike-complexity density is not the induced ten-dimensional volume element.**

   The manuscript obtains equation (4.4) by setting $\mu = \sqrt{FH}$ for the effective two-dimensional metric. This produces the radial/angular dependence

   $$\mu_{\text{paper}} \propto \frac{r\sqrt{K_y}\, G^{3/2}}{\sqrt{r^2+\ell^2}}.$$

   That operation is not valid for the claimed codimension-one volume. Taking the determinant directly on the fixed-y hypersurface of the ten-dimensional metric (2.1), with coordinates

$(t, r, \theta, \phi_1, \phi_2, T^4)$, gives instead

$$\sqrt{|h_9|} \;\propto\; \frac{r\sqrt{K_y(r,\theta)}\,G(r,\theta)}{\sqrt{r^2+\ell^2}}\sin\theta\cos\theta,$$

where the omitted factor depends only on the charges and the coordinate volume of T^4. In particular, for $Q_1 = Q_5 = 1$ the difference is an extra factor $\sqrt{G}$ in equations (4.4)–(4.5).

The effective metric in equation (2.19) was engineered so that a curve length packages the compact directions of a wrapped codimension-two area. Taking the determinant of its (t, r) block packages that internal factor a second time; it cannot be assumed to yield a wrapped codimension-one volume. The error changes precisely the core weighting that is central to the paper. It propagates through the primitive and subtraction in equations (4.13)–(4.30), the large-radius coefficients, the small-T formula, and all exact complexity minima and plots in Figures 15 and 18–22. The authors must rederive the hypersurface measure from (2.1), state which region and signature are intended, and repeat the complete complexity analysis. Alternatively, if $\sqrt{FH}$ is meant to define a new reduced diagnostic rather than a ten-dimensional CV volume, the paper must say so and supply a physical justification for that diagnostic.

3. **The small-r regulator is part of the claimed effect, and some selected branches fail the manuscript's own reality condition.**

Equation (3.15) requires $F(r;\theta_0) < F_0$ throughout the timelike radial interval. Substitution of the parameters in equation (3.69) and the selected values printed in Figure 8 shows that this is not true as $\epsilon \to 0$. For $x_E = 0.2$, the cap limit is $F(0;\theta_0) = \ell_2^2 = 0.2243034$ in the units used in the plots. At $T = 12$, Figure 8 gives $\theta_0^* = 0.758335$ and $r_0^* = 0.116926$, for which $F_0 = 0.159177$. The inequality is reversed below approximately 9.9×10^{-4}. The printed $T = 10$ saddle likewise has a forbidden core below approximately 1.9×10^{-4}, and the $T = 8$ candidate has $F_0 < \ell_2^2$ as well. A regulator near 10^{-3} can hide the first failure, but it does not define a real branch in the limit claimed by the construction.

There is a second, independent infrared problem on the horizon-label side. Equations (3.75a)–(3.75b) and (A.33) imply $K_{\rm tim}, K_{\rm sp} \sim c_0/r$ as $r \to 0$ for $\theta_0 > \theta_\star$. In the physical cap-angle part of the lift, $\theta < \theta_\star$, $F(r,\theta)$ tends to a nonzero constant and H is finite. Hence equations (3.61a)–(3.61b) give

$$\Delta_b(r,\theta) \sim -\frac{F_{\rm cap}(\theta)c_0^2}{r^2}, \qquad \sqrt{\Delta_b + i0} \sim \frac{ic_0\sqrt{F_{\rm cap}(\theta)}}{r}.$$

Both equations (3.89) and (3.90) therefore contain an imaginary $\log(1/\epsilon)$ contribution. The statement following equation (3.89) that the timelike contribution is finite addresses only the absence of a large-r UV divergence; it overlooks this lower-end divergence. Adding the spacelike and timelike pieces in equation (3.97) does not cancel it. Figure 11 places some selected small-T saddles on the $\theta_0 > \theta_\star$ side, so the displayed imaginary area is directly affected.

Appendix A itself shows why the issue is unavoidable: the enhancement in equations (A.46)–(A.49) grows logarithmically as the crossover scale tends to zero and must saturate when that scale reaches ϵ. Thus the height and width of the peaks in Figures 5 and 17, and potentially the selected angles, depend on the order of the limits $\theta_0 \to \theta_\star$ and $\epsilon \to 0$. The authors must formulate a regular cap/horizon continuation, enforce equation (3.15) before minimization, identify and subtract any legitimate infrared divergence, and show convergence as ϵ is lowered. At present the strongest “transition-region” signal may be regulator controlled.

4. **The branch gluing and complex-saddle selection need a variational and topological prescription.**

   Equation (3.13) imposes $\Pi_{\rm sp}^2 = F_0$, and equation (3.82) ensures that the two coordinate-time profiles meet at $r = \epsilon$. Neither step is a junction condition derived by varying a piecewise surface. On the cap side, equations (3.72)–(3.73) give finite but generically unequal tangents and canonical momenta, so the joined curve can have a corner at a regular cap. On the horizon side, Schwarzschild-like coordinates are singular and the matching must be analyzed in regular coordinates. The angular lift also spans sectors in which the same $r = 0$ locus has cap and horizon interpretations. Continuity of t alone does not establish smoothness, extremality, homology, or an allowed continuation through these sectors.

   Likewise, the principal square-root rule in equation (3.88) specifies a local branch of the integrand but not the global complex-saddle problem. Equations (3.64) and (3.107) select the smallest real part of a complex area. A physical complex saddle normally follows from an analytic continuation and an integration contour, with possible Stokes transitions; “minimum real part” is not by itself such a prescription. The authors should derive transversality conditions at the junction, formulate the surface in coordinates regular at the relevant endpoint, state the homology/topology condition, and explain which complex saddles lie on the appropriate contour. These points are prerequisites for interpreting the branch folds as physical saddle transitions.

5. **There are normalization inconsistencies in the exact area and in the effective metric.**

   Equations (3.89)–(3.90) integrate one radial copy of the upper timelike and spacelike branches, although the lower branches are separately obtained by time reflection. The missing reflected copy is visible internally: the asymptotic divergence is $\mathcal{N}Q\log R$ in equation (3.41), whereas the exact angular lift gives only $\mathcal{N}Q\log R/2$ in equation (3.92). The factor 1/2 is exactly the integral $\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$ without an overall factor two for the two reflected pieces. The same mismatch occurs between the exact and asymptotic finite areas. If the exact expression is intended to be a half-surface quantity, the entropy definition and all comparisons must be changed consistently; otherwise equations (3.60) and (3.89)–(3.99) need an overall factor of two. Although a common factor would not move an ideal minimum, it changes all stated entropy normalizations and the claimed short-interval match.

   In addition, equation (2.19) appears to mistranscribe the effective metric of Ref. [6]. With the manuscript’s definition $Q = \sqrt{Q_1Q_5}$, the cited expression contains $Q_1Q_5 = Q^2$ in the corresponding places, while (2.19) uses Q. Consequently the general charge powers in F and H in equation (2.23) are also changed. Setting $Q_1 = Q_5 = 1$ hides the issue numerically, but the analytic normalization and dimensions should be corrected throughout. The compact factor $\mathcal{N}$ also disappears without explanation in equations (4.60)–(4.63), even though those equations descend from densities proportional to $\mathcal{N}$.

6. **Equations (3.93) and (3.94) are not equivalent renormalizations.**

   Equation (3.93) subtracts the universal large-R logarithm. Equation (3.94) instead subtracts $\sqrt{H(r,\theta)}$ from ϵ to infinity. Although $\sqrt{H} = Q/r + O(r^{-3})$ in the UV, extending this subtraction through the localized interior removes an additional finite, geometry-dependent and generally ϵ-dependent term. The statement that the finite localized contribution is kept is therefore not correct as written. The extra term may be common to all candidates in one fixed background and hence leave a branch minimizer unchanged, but it changes the reported real area, comparisons between backgrounds, and matching to the asymptotic calculation.

The authors should use a covariant boundary counterterm or restrict the integrand subtraction to an explicitly asymptotic region and add back its finite primitive. If equation (3.94) is retained as a different scheme, that scheme and the finite conversion to equation (3.93) must be stated. The same level of care is needed for the signed volume subtraction in equations (4.25)–(4.30), especially because the quantity plotted there can be negative and its minimization is sensitive to finite conventions.

7. **The independent minimization of volume is not the standard subregion-CV prescription.**

   Equations (4.72) and (4.81) choose a surface by independently minimizing a finite spacelike-minus-timelike volume. Standard subregion CV instead evaluates the volume of the region bounded by the relevant HRT or entropy surface; it does not normally select a different boundary by minimizing the renormalized volume itself. Ref. [42] motivates a timelike volume enclosed by specified extremal branches, but does not establish the new selection rule used here. Moreover, equation (4.30) is a signed finite difference, not a positive local volume functional whose stationary points follow from a maximal- or minimal-volume equation.

   The manuscript must derive this rule from a boundary proposal or a bulk variational principle. The more conservative calculation would evaluate the corrected volume on the area-selected branch, using the same black-pole parameters. At present, the paper calls the two quantities complementary probes of the same branch geometry while using different selection rules and even different backgrounds: equation (3.69) uses $x_E = 0.2$ for entanglement, while equation (4.65) uses $x_E = 0.6$ for complexity. A matched-x_E comparison is essential.

8. **The large-radius checks do not establish the exact short-interval limit.**

   The exact branches in equations (3.67)–(3.82) extend to the lower endpoint $r = \epsilon$ for every r_0. Section 3.2 instead imposes a large-radius window with an arbitrary lower edge $r_* = 10$. It is not shown that the part of the exact surface below r_* cancels or becomes negligible as $T \to 0$, particularly after lifting over $\theta \neq \theta_0$, where the reduced near-null cancellation need not persist. Thus equations (3.52)–(3.56) are useful calculations in a truncated asymptotic geometry, but they are not yet checks of the exact prescription. The analogous complexity result in equations (4.60)–(4.63) contains r_* in the subleading logarithm and inherits the incorrect volume density noted above.

   A matched asymptotic expansion should show how the core and exterior regions combine, or the exact numerical answer should be driven far enough into the small-T regime to demonstrate the coefficient and constant in equation (3.56), independence from r_*, and the expected boundary-CFT or BTZ limit. The BTZ calculation in Section 4.1 does not by itself validate the localized observable: the BTZ region includes an interior contribution, whereas the localized construction is explicitly an exterior-patch gluing.

9. **The numerical evidence is not reproducible or demonstrably converged.**

   The paper does not state ϵ, $R_{\max}$, the radial and angular grids, root-finding and minimization tolerances, the treatment of turning points and sign changes in Δ, or error estimates. These omissions are decisive near $\theta_\star$, where Appendix A predicts logarithmic behavior and the admissible windows can be narrower than a sampling cell. Figures 5 and 17 contain jagged spikes and repeated or isolated values in the displayed $T_{\max}(\theta_0)$ data. Figures 11 and 21 show abrupt jumps and staircases in selected angles that are not identified as physical phase transitions or tested against a finer grid.

There are also discrepancies between plots that purportedly show the same selected solutions. At $x_E = 0.6$ and $T = 0.5$, Figure 19 labels the selected point by $\theta_0 = 0.6920$, $r_0 = 3.9870$, and $C = 0.0168$, whereas Figure 22 uses the same θ_0, T, and C but gives $r_0 = 4.1227$. Smaller differences occur at $T = 2$ and $T = 4$. Moreover, the complexity saddle moves from near $\theta_\star \simeq 0.693$ to approximately 0.62–0.59 at larger T in Figure 19, which is movement away from, rather than toward, the transition angle despite the surrounding interpretation.

The revised analysis should provide a single reproducible data set and code, regulator scans, grid and tolerance scans, residuals for the fixed-T equation, checks of equation (3.15) at every quadrature point, and uncertainty bands or tables for all selected quantities. Candidate minima on both radial branches should be resolved independently, and every apparent jump should be classified as a converged saddle exchange or a numerical artifact.

Novelty and significance

Applying timelike observables to an analytic, internally localized D1–D5 black hole is a worthwhile direction. The distinction between the branch label θ_0 and the physical lift angle θ, the folded fixed-interval problem, and the local crossover analysis are potentially novel. If a valid extremal prescription exhibits a robust preference for the cap/horizon transition, that would be interesting to the holography and microstate-geometry communities.

At present, however, the physical significance cannot be separated from the ansatz and regulator. The principal surface is a restricted family not shown to be stationary; the transition enhancement runs into the lower cutoff; and the complexity measure has the wrong internal warp-factor weight. Consequently the plotted location of a preferred branch, the complex area, and the finite volume cannot yet be claimed as observables of the black-pole geometry. The novelty would become assessable only after the corrected construction is compared with BTZ and with the boundary short-interval expectation at the same charges and energy.

Minor comments

1. The text near equation (3.88) says that K_y and G can change sign. The warp factors in the displayed physical geometry are positive; it is $\Delta_{\rm sp}$ or $\Delta_{\rm tim}$ that changes sign. This should be corrected because the sign is central to the real/imaginary decomposition.

2. The sentence following equation (3.68) appears to associate a logarithmic $R_{\max}$ dependence of the spacelike time integral with the area subtraction. The time integral is UV convergent; the logarithm belongs to the area. The discussion should distinguish the two cutoffs and the two integrals.

3. The derivation of the coefficient C_μ in equation (4.21) should display the required $O(r^{-4})$ expansions of both K_y and G. Equation (2.24), as printed, does not contain enough information

by itself to determine this coefficient. Equation (4.60) should also separate the $r_0^{-2} \log r_0$ and nonlogarithmic r_0^{-2} orders clearly.

4. Several multipanel figures, especially Figures 16–18, have blank or minimal subcaptions. Each numerical caption should state all physical parameters, cutoffs, normalization conventions, branch colors, and sampling information. The selected values in profile figures should be generated from exactly the same data records as the minimization figures.

5. The comparison with the literature should more clearly distinguish the restricted localized-RT estimate in Ref. [6], the analytic-continuation definition of timelike entanglement, and the proposed timelike-volume construction. These are not interchangeable extremization principles.

Required scope of a new submission

A publishable version would need, at minimum: (i) a ten-dimensional variational derivation or an explicit and quantitatively controlled approximation statement; (ii) a corrected induced volume element and a complete recomputation of complexity; (iii) regular junction, homology, and complex-contour prescriptions; (iv) a consistent normalization and renormalization scheme; (v) regulator-independent, convergence-tested numerics with code or data; and (vi) same-background comparisons of the two observables and their BTZ or boundary limits. These changes could preserve some of the transition-region analysis, but the principal physical claims must be re-established rather than merely rephrased.