Referee Report

on "Notes on the bootstrap of four-point conformal integrals"
arXiv:2607.11645

Summary

The manuscript develops a bootstrap strategy for finite four-point conformal integrals in position space, with emphasis on integrals obtained by projecting three- and four-loop f-graphs onto four external points. The proposed workflow combines leading-singularity information, decompositions into pieces with simpler rational leading singularities, parity-restricted single-valued multiple-polylogarithm ansätze, collision-limit data obtained by expansion by regions and integration-by-parts reduction, and constraints from permutations of the external points. A further ingredient, called the "mirror constraint," is obtained by interchanging the two algebraic expressions for z and $\bar{z}$ in a series originally derived in a specified hierarchy.

The paper first organizes fifteen inequivalent three-loop conformal integrands. Fourteen are evaluated using `HyperlogProcedures`, while the remaining one is related to the others through a Gram-determinant identity. Magic identities and external permutations reduce these to twelve functions. At four loops the author enumerates 412 projected integrands, of which 164 are reported to be directly calculable with `HyperlogProcedures`. The bootstrap is then illustrated on selected cases. The previously studied $\mathcal{I}^{(4)}_{173}$ serves as the principal test of the mirror constraints, and new analytic expressions are claimed for pieces of $\mathcal{I}^{(4)}_{176}$ and $\mathcal{I}^{(4)}_{181}$. The paper also proposes an empirical relation between generalized boxing or graph reduction and the possible appearance and entry positions of additional letters in the svMPL alphabet.

There is potentially useful material here. The broad catalogue, the inclusion of nonplanar position-space sectors, the practical use of distance Gram determinants at numerator-bearing vertices, and the attempt to isolate single-leading-singularity pieces are all interesting. Several algebraic checks are also encouraging. In particular, the decompositions displayed for $I^{(4)}_{176}$ cancel correctly when their numerators are recombined, and the decomposition of $I^{(4)}_{181}$ is consistent with equation (3.17). The three-loop sector supplies a valuable benchmark, while the agreement reported for the known $\mathcal{I}^{(4)}_{173}$ result is a useful check of the implementation.

At present, however, the central four-loop conclusions are not established at the standard required for publication in a journal such as JHEP or PRD. The basic integral normalization is not defined, the mirror prescription is not mathematically justified, completeness of the cut and function-space analyses is not demonstrated, and the new weight-eight expressions receive only rather limited independent validation. There are also definite algebraic and sign errors in important displayed equations. My recommendation is therefore **major revision**. I would be open to publication after the points below are addressed and the new analytic results are certified by substantially stronger checks.

Major comments

1. **The integrated objects and their normalization must be defined unambiguously.**

The manuscript distinguishes a rational integrand $I_i^{(L)}$ from an integrated function $\mathcal{I}_i^{(L)}$, but it never gives a defining formula for the latter. A reader needs to know the integration measure, factors of π, whether the integration is initially in four dimensions or regulated, the Euclidean contour, and the external conformal prefactor that is removed before the result is denoted by a function of $z, \bar{z}$. The displayed four-point integrands have conformal weight one at every external point. After integrating the internal vertices, the covariant object still has length dimension -4; a factor such as $x_{13}^2 x_{24}^2$ is therefore needed to define a dimensionless function of u and v. This convention is essential when comparing different external permutations and when stating the invariances in equation (2.39).

The missing convention also creates a concrete dimensional problem in the cut formulas. From equation (2.4), λ_{1234} is the square root of a determinant of squared distances and hence has length dimension four. With the cross-ratio conventions of the paper one has, up to the branch sign,

$$\lambda_{1234} = x_{13}^2 x_{24}^2 \, (z - \bar{z}).$$

Equations (2.14) and (2.19) effectively identify λ_{1234} with the dimensionless quantity $z-\bar{z}$. This is valid only after an explicitly stated external rescaling. The same issue affects the quoted leading singularities and the definition of the coefficient functions in equation (3.11). The revision should begin with a precise equation defining $\mathcal{I}_i^{(L)}(z, \bar{z})$ and should carry this normalization consistently through the cuts, permutations, magic identities, numerical comparisons, and supplementary expressions. The normalization of the two-point family $G[\ldots]$ in equations (2.37)–(2.38) should be specified at the same time.

2. **The mirror constraints require a derivation in terms of exact analyticity or diagonal regularity.**

   This is the most serious conceptual issue. Section 2.3 first constructs a series with $|\bar{z}/z| < 1$, truncates that series, and then obtains additional equations by substituting the opposite hierarchy for z and $\bar{z}$. An asymptotic expansion is not in general valid after such a substitution. Discarded terms of arbitrarily high order in $\bar{z}/z$ can become enhanced in the exchanged hierarchy and can contribute to precisely the powers of u and $Y = 1 - v$ used in the matching. The manuscript itself calls this use "inconsistent" or "illegal," but the absence of a physical singularity at $z = \bar{z}$ does not by itself prove that the truncated continuation is valid.

   There is a second way to see the problem. The parity-projected ansatz in equation (3.11) is already invariant under the appropriate exact exchange of z and $\bar{z}$ term by term. An exact exchange cannot create independent linear information. The extra rank reported in Section 3.2.1 must therefore come from the nonuniform truncation, and it appears to be an indirect attempt to impose regularity as the two roots coalesce. The rational example in equations (2.43)–(2.48) illustrates a legitimate condition: the numerator must be divisible by $(z-\bar{z})^2$ so that the apparent diagonal pole cancels. It does not establish that the same condition can be extracted from an off-domain truncation of an infinite svMPL series.

   The author should reformulate this step. One acceptable route would be to derive exact Taylor or divisibility conditions at $z = \bar{z}$ for every rational prefactor and impose them directly on the single-valued functions. Another would be to analytically continue the relevant iterated integrals with full control of the branch and remainder terms, and then prove that the retained orders close under the exchange. In either case, the paper should explain why no omitted term can modify the linear constraints. The successful recovery of the already known $\mathcal{I}_{173}^{(4)}$ is an important benchmark, but it cannot substitute for this derivation when the same prescription is used to claim new functions.

3. **The leading-singularity classification and alphabet assumptions are not shown to be complete.**

   The paper displays informative cuts, but its central claims concern the full set of analytic structures of an integral. For $I^{(4)}_{176}$, the text asserts that only the two stated rational leading singularities occur; for the pieces of $I^{(4)}_{181}$, it similarly claims a single type of leading singularity after the numerator decomposition. No exhaustive list of independent maximal and submaximal cuts, residue contours, branch choices, or higher-order poles is provided. Cancellation on selected cuts does not rule out additional rational residues, double-pole sectors, or an elliptic cut on a different contour. This matters directly because the prefactor and function space of the bootstrap are chosen from this analysis.

   The generalized-boxing argument is likewise suggestive rather than complete. Appendix A shows that acting with the Laplacian can also produce same-loop terms when a numerator generates a double pole, so graph reduction is not in general an exact descent to a single lower-loop function. Appendix B explicitly calls the proposed letter rule empirical. The subsequent success of a bootstrap performed within the alphabet selected by that rule is not an independent proof that the alphabet is complete: a too-small ansatz can fit a finite set of boundary coefficients while missing terms that vanish to the tested order.

   For the principal new examples, the revision should therefore provide a machine-checkable cut table or another completeness certificate. It should also test stability under a systematically enlarged alphabet and under relaxed entry restrictions. A differential equation, a symbol/coaction analysis, a $d\log$ representation, or an independent direct evaluation of a sufficiently rich set of cuts would be particularly valuable. If a proof is not available, the status of the $I^{(4)}_{176}$ and $I^{(4)}_{181}$ formulas should be stated as a well-tested conjecture rather than an exact evaluation.

4. **The expansion-by-regions argument needs complete power counting and regulator bookkeeping.**

   The collision limits in Section 2.2 supply nearly all of the data used to fix the ansatz, so their derivation deserves more detail. The sentence following equations (2.33)–(2.35), according to which the nth numerator term is simply of order Y^n, is not generally true in a soft region. If $x_\ell = \sqrt{u}\, q$ and $x_1 = 0$, then, for example,

   $$2x_\ell \cdot x_3 - x_{\ell 1}^2 = 2\sqrt{u}\, q \cdot x_3 - uq^2,$$

   so tensor reduction produces mixed powers of u and Y. A joint power-counting argument is needed to prove that the stated truncations through $\mathcal{O}(u^0, Y^3)$ or $\mathcal{O}(u^0, Y^4)$ retain every term that can feed the matched coefficients. In real Euclidean kinematics, moreover, $u = x_2^2$ while $Y = 2x_2 \cdot x_3 - u$, so generically $Y = \mathcal{O}(\sqrt{u})$. The sequential limit used in the calculation should be defined as an analytic limit and its relation to the physical Euclidean region explained.

   There is also a definite error in equation (2.35):

   $$x_{b1}^2 - \left(x_{b1}^2 - x_{a1}^2\right) = x_{a1}^2,$$

   not x_{ab}^2. For $x_1 = 0$, the intended identity is

   $$x_{ab}^2 = x_{b1}^2 - \left(2x_a \cdot x_b - x_{a1}^2\right),$$

which is the expression used by the following geometric expansion. This should be corrected.

Finally, individual regions and IBP coefficients contain poles in the dimensional regulator. The paper should state the maximum pole order produced by the reduction, the required ϵ depth of every two-point master, and show cancellation of the poles in the finite conformal integral. The relation between the `HyperlogProcedures` convention $d = 4 - \varepsilon$ and the paper's $d = 4 - 2\epsilon$ must be incorporated in the defining formulas. Without these data it is impossible to verify that the master expansions in equation (2.38) are deep enough for the finite coefficient.

5. **The decomposition and notation for $I^{(4)}_{176}$ are internally inconsistent.**

   Equation (2.26) introduces a two-part decomposition. Equation (2.27) then introduces an α-dependent three-part decomposition, using overlapping labels for objects that are no longer the same. Equation (2.31) treats all three pieces, while the paragraph below it refers to taking $\alpha = 0$ in equation (2.32), although α occurs in equation (2.27). Section 3.2.2 states that the original integral has been split into three parts but then writes $\mathcal{I}^{(4)}_{176} = \mathcal{I}^{a}_{176} + \mathcal{I}^{b}_{176}$ while also discussing a nonzero $\mathcal{I}^{c}_{176}$. The footnote mapping these objects to basis entries 117 and 120 further allows unspecified permutations and constant factors. This is too ambiguous for one of the main new results.

   The author should choose a value of α, define each rational integrand on one line with nonoverlapping notation, and give one exact recombination formula both before and after integration. A table should map each piece to its basis identifier, external-point permutation, conformal prefactor, and ancillary-file name. The numerical checks should then be quoted separately for all pieces and for their recombined sum. It would also help to explain why this particular decomposition is preferred among the continuous family generated by α.

6. **Several displayed relations contain definite errors or missing factors.**

   These are not merely typographical in the present context, because the paper uses such identities to organize and validate the bootstrap.

   - The first two magic identities in equation (3.2) have incorrect minus signs. With the basis definitions used in the paper, the identities relate the corresponding positive Euclidean integrals with a plus sign. Equivalently, the associated linear relations are differences such as $I_2(2,1,3,4) - I_3(1,2,3,4) = 0$, not sums. The same correction applies to the I_9/I_{11} relation. All uses of these identities in the count of independent three-loop functions should be rechecked.
   - Equation (2.12) drops the factor λ^{-1}_{1234} that is present in equation (2.8), although the text describes the remaining cut as the same one. A rescaling of the elliptic-curve coordinate can change the presentation of the square root but does not silently remove an external prefactor. The coordinate rescaling relating equations (2.9) and (2.11) should be given explicitly.
   - Equations (3.13), (3.15), and (3.16) use calligraphic $\mathcal{I}$ for bare rational integrands, contrary to the distinction stated earlier in the paper. This should be changed to I or accompanied by the integration operation.
   - The symbol Y denotes both the coordinate on the elliptic curve in equations (2.9) and (2.11) and the cross-ratio variable $Y = 1 - v$ from Section 2.2 onward. These should be separated.

After correcting these points, the author should rerun every analytic and numerical identity from the corrected definitions and include a concise list of the checks. In particular, the three-loop catalogue should be regenerated after fixing equation (3.2), rather than assuming that the final count is unchanged.

7. **The new analytic results need a self-contained presentation and substantially stronger validation.**

   The principal new weight-eight expressions are not displayed, summarized by a symbol, or characterized by a differential equation in the article. The raw supplementary expressions use notation such as `I[z,...]`, `zz`, and `f[3]` without a complete legend. The reversal between the word order in equation (2.28), the "last entries" described around equation (2.32), and the ordering in the machine representation is also not made transparent. As a result, a reader cannot unambiguously reconstruct the functions that are being claimed.

   The paper should provide at least a compact exact characterization of every new answer: the rational prefactors, alphabet and word convention, parity, first- or last-entry restrictions, coefficient-vector length, normalization, and the formula for assembling the pieces. A machine-readable manifest should identify which files are genuinely new and which reproduce known results. The supplied data should be sufficient to evaluate each function at arbitrary Euclidean points without relying on undefined local conventions.

   The current numerical tests at two kinematic points, with typical relative precision of only 10^{-4}–10^{-5}, are useful sanity checks but are not adequate to certify long weight-eight expressions obtained from a restrictive ansatz. The revision should include more points in distinct kinematic regions, higher-precision integration, checks of every relevant external permutation, and tests of additional collision coefficients that were not used in fitting. An independent differential equation or direct integration would be much stronger. The known $I^{(4)}_{173}$ result should remain as a benchmark, while the genuinely new $I^{(4)}_{176}$ and $I^{(4)}_{181}$ results should receive their own independent checks.

8. **The scope, novelty, and physical significance should be stated more sharply.**

   The basic leading-singularity/svMPL/boundary-data bootstrap follows the three-loop strategy of Drummond and collaborators cited in the Introduction. The graphical-function and `HyperlogProcedures` framework is also an established body of work, and $\mathcal{I}^{(4)}_{173}$ was bootstrapped in the 2025 paper cited in Section 3.2.1. The present paper's apparent new content is therefore the 412-integrand classification, the numerator and Gram-determinant techniques used in the cut analysis, the empirical graph-reduction rule for letters, the mirror prescription, and the results for $I^{(4)}_{176}$ and $I^{(4)}_{181}$. This division between prior method, benchmark, and new result should be stated explicitly.

   The paper would benefit from a table listing, for each featured integral, whether its integrand, function space, analytic value, and numerical validation are new. The relation to earlier calculations of four-point graphical functions should be discussed integral by integral, rather than only at the level of general methodology. It should also be explained whether the new functions enter a complete correlator or amplitude observable, or whether their present significance is primarily as individual master-like conformal integrals. In my view the latter is still worthwhile, especially for the nonplanar sectors, but the physical payoff should not be overstated before the new answers are incorporated into a gauge-invariant observable.

9. **The three- and four-loop catalogues need a reproducible definition of equivalence and independence.**

The numbers fifteen and 412 are useful only if the precise quotient is clear. The manuscript should provide the generating f-graphs, the four-point projection rule, the canonicalization under graph automorphisms and external permutations, and a reproducible script that produces the list. It should distinguish a catalogue or spanning set of projected integrands from a basis of integrated functions: magic identities, permutation relations, and four- dimensional Gram relations show that these notions are not identical.

The four-loop Gram discussion near equation (3.8) also needs a consistent degree convention. Ordinary polynomial degree requires the coefficients of the 5×5, 6×6, and 7×7 Gram minors to supply respectively three, two, and one further powers when constructing a cubic numerator; the text instead quotes negative degrees, apparently using a conformal-weight convention. The convention should be defined, and the rank and nullity of the resulting system should be reported. Providing the canonical representative, orbit size, and known relations for each entry would make the catalogue a lasting and independently usable result.

Minor comments

1. Equation (2.28) defines $\mathrm{L}_{a_1 \dots a_n} = \mathrm{I}_{z,a_n,\dots,a_1,0}$, so the word order is reversed. The subsequent statements about a letter appearing in the "last two entries" should specify whether they refer to the L word or the I indices.

2. Section 2.1.4 should reconcile the stated number of ansatz elements with the dimensions used in the supplied examples. More generally, every example should list the dimension before constraints, the rank supplied by each of the six boundaries, the additional rank attributed to mirror conditions, and the final nullity.

3. In equation (3.10), $I_{49}^{(4)}$ appears twice. If this is intentional because 32 projections yield 31 distinct functions, that fact should be stated at the equation rather than left implicit.

4. The phrase that finite conformal integrals have no pole or branch cut on the real line should exclude the singular points $z = 0, 1, \infty$ and should specify the sheet. These qualifications are relevant to the proposed diagonal regularity argument.

5. The term "rigid" is used in a nonstandard way for functions expected to have uniform transcendental weight. It should be defined precisely or replaced by "pure/uniform-weight" where that is what is intended.

6. The first master in equation (2.37) appears with a different bracket structure from the others. The topology and propagator ordering of all four masters should be shown graphically or defined in a uniform family notation.

7. The numerical tables should quote absolute as well as relative errors, the integration precision requested, and the uncertainty of the numerical integrator. Agreement should be judged against that uncertainty rather than by the number of displayed digits alone.

8. The supplementary code should be associated with a versioned release and state software and package versions. A manifest connecting each manuscript equation to the relevant input and

output would considerably improve the value of the results to the community.

9. There are numerous grammatical slips and encoding artifacts, including malformed occurrences of "ansätze." A careful language and notation pass is needed after the technical revision.

Recommendation

I recommend **major revision**. The manuscript addresses a worthwhile problem and contains promising constructions, but the new four-loop evaluations are not yet supported as exact results. A publishable revision should, at minimum, define the normalized conformal integrals, correct the concrete equation errors, replace the mirror prescription by a controlled analyticity argument, demonstrate the relevant cut and alphabet completeness, document the region/IBP calculation, and subject the new $I_{176}^{(4)}$ and $I_{181}^{(4)}$ expressions to independent high-precision or analytic checks. If these points are resolved, the catalogue and the nonplanar examples could make a useful contribution to the literature on conformal integrals and graphical functions.