Referee Report

"Modular structures in the DSSYK partition function"

arXiv:2607.11828

Summary and overall assessment

This paper studies the disk partition function of double-scaled SYK (DSSYK) in the joint low-temperature and semiclassical regime. Starting from the exact Bessel-function representation

$$F(x) = \frac{2}{x}\sum_{n\geq 0}(-1)^n(2n+1)I_{2n+1}(x)q^{n(n+1)/2}, \qquad q = e^{-\lambda},$$

the authors organize the dominant large-x expansion in the differential ring generated by the Eisenstein series E_2, E_4, E_6 and the factor $(q;q)_\infty^3$. Modular inversion then reorganizes the result as a small-λ transseries involving

$$\widetilde{q} = e^{-4\pi^2/\lambda}, \qquad y = \frac{24\pi^2}{\lambda\beta}, \qquad \Lambda = e^{y/3}\widetilde{q}.$$

For the logarithm of the partition function, the authors give closed expressions through order λ^2 for the coefficient of every proposed $\widetilde{q}^n$ sector, resum those coefficients in terms of Eisenstein series evaluated at Λ, and derive an exact differential equation, equation (6.3), that connects the modular derivative to derivatives with respect to the Bessel argument. Finally, they argue that the dominant-branch expansion of Z, unlike that of $\log Z$, is supported on triangular powers and compare these exponents with winding saddles in the bilocal-Liouville description.

There is a valuable paper here. In particular, the operator representation (4.20), the quasi-modular form of the coefficients in (4.22), the subleading $O(\lambda)$ and $O(\lambda^2)$ corrections, and the exact equation (6.3) provide an elegant synthesis. Several central checks are convincing. The product in the Bessel asymptotic indeed becomes $32\mathcal{D} + 3 - 4j(j-1)$ on a term $q^{n(n+1)/2}$; Bessel's equation directly gives (6.3); the chain rule then gives (6.6); and the divisor sums in (5.37) correctly turn the proposed all-n expressions into (5.38). These results should be useful to readers working on DSSYK, sine-dilaton gravity, and semiclassical transseries.

However, the present version overstates both the proof and the physical scope of its principal non-perturbative claims. The arbitrary-n formula (5.35), on which the closed resummation (5.38) rests, is inferred from finite-order computation rather than proved. More importantly, the analysis retains the dominant e^x Bessel branch while the claim that it exhausts the nonperturbative saddle spectrum ignores the upper-edge branch associated with the omitted relative e^{-2x} contribution. The novelty relative to Ref. [27] also needs to be stated more accurately. These are not merely stylistic matters, because they concern what has actually been established and what object is being matched to the bulk saddles.

I therefore recommend **major revision**. I would be favorable toward publication after the authors supply an all-n derivation (or explicitly downgrade the result to a conjecture), restrict or complete the saddle-spectrum claims, and correct the internal normalization errors listed below.

Major comments

1. **The claimed all-n result in equation (5.35) needs a proof.**

The text states that the three functions $f_{n,0}$, $f_{n,1}$, and $f_{n,2}$ were computed for $n \leq 20$ and that the general expressions (5.35) were then found. Appendix A does not prove this statement for arbitrary n: it isolates the coefficient of $\widetilde{q}$ and derives only the $n = 1$ functions displayed in (5.29). Saying that the procedure can be extended does not establish that, for every n, the answer reduces precisely to the combination of $\sigma_1(n)$ and $\sigma_3(n)$ shown in (5.35), with no additional divisor convolutions.

This gap is central because (5.38) is obtained by summing (5.35) over all n. The summation from (5.35) to (5.38) is algebraically consistent, but it is conditional on the unproved extrapolation. The authors should derive a recursion for a general $\widetilde{q}^n$ mode and solve it through $O(\lambda^2)$, or directly prove the generating-function identity (5.38) before extracting its coefficients. Substitution of the proposed generating function into the exact differential equation, supplemented by sufficient boundary data, might provide an economical route. At minimum, the finite symbolic procedure and reproducibility code should be supplied. If an analytic proof cannot be given, (5.35) and (5.38) must be labeled as conjectural identities checked through $n = 20$, and the abstract and conclusions must be weakened accordingly.

2. **The dominant modular sector does not exhaust the nonperturbative saddle spectrum.**

   The large-x expansion used in (4.20)–(4.22) is the e^x branch of the modified-Bessel asymptotic. The authors correctly observe after (4.24) that its Borel completion also contains a relative contribution of order e^{-2x}. This is not unrelated debris: in the exact low-temperature partition function it is the contribution based on the upper spectral edge, proportional to e^{-x} rather than e^x. Ref. [27] displays both terms in its equation (3.27), and Ref. [29] treats both the lower- and upper-edge saddle families in its equations (3.33a) and (3.33b), as well as in equations (C.10) and (C.11).

   Section 5.3 proves, at most, triangular support for the formal $\widetilde{q}$ sectors generated by the dominant large-x branch. It therefore does not justify the statements in the abstract and Section 7 that the "entire non-perturbative sector" of Z is supported there, that the comparison "exhausts the saddle spectrum," or that no exponents remain to be accounted for. The omitted e^{-2x} relative sector supplies precisely another branch of the exact answer and another saddle family. The authors should either incorporate that branch and perform the corresponding modular/saddle analysis, or consistently state that their result concerns only the dominant lower-edge branch and its $e^{-4\pi^2/\lambda}$ sectors. The latter restriction would preserve the main mathematical result while removing an incorrect physical conclusion.

3. **Equation (7.1) is a leading low-temperature action match, not a match of the new subleading data.**

   The subtraction leading to (7.1) correctly reproduces the displayed lower-edge Schwarzian action difference at the order retained in equation (3.33a) of Ref. [29]. Nevertheless, the underlying saddle parameter is itself given as a large-β expansion, and the full DSSYK saddle action contains further terms beginning schematically at $1/(\lambda\beta^2)$. In the scaling used for (5.24)–(5.38), where $\lambda\beta$ is fixed, such terms are $O(\lambda)$. They therefore occur at exactly the order at which the new functions $f_{n,1}(y)$ begin. Equation (7.1) establishes the leading instanton action in the lower-edge Schwarzian limit; it does not yet demonstrate that the $O(\lambda)$ or $O(\lambda^2)$ functions agree with corrections computed around those saddles.

   The paper should make this precision explicit and distinguish three statements: equality of the leading exponential action, equality of the fluctuation series multiplying that exponential, and equality of the full finite-λ sector. Only the first is shown. A comparison of the subleading saddle expansion with (5.35) would be an important additional result, but absent such a calculation the stronger wording should be removed.

There is also an internal mismatch in the proposed future comparison of one-loop weights. Section 5.3 correctly explains that saddles must be compared with Z, not with $\log Z$. Yet the last paragraph discussing the comparison proposes using $C_n = f_{n,0}(0)$, which are coefficients of $\log Z$ and are nonzero for every integer n. At leading order one instead has

$$\frac{Z}{Z_{\text{pert}}} = (\Lambda;\Lambda)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)\Lambda^{k(k+1)/2}.$$

The coefficients $(-1)^k(2k+1)$ at triangular exponents are the quantities to compare with determinants, phases, negative modes, and thimble weights. Since Ref. [29] already gives one-loop saddle contributions, the paper should at least formulate this comparison correctly; carrying it out would substantially strengthen the claimed bulk interpretation.

4. **The novelty relative to Ref. [27] is described too broadly.**

   The introduction characterizes the earlier result essentially as the β-independent factor $\eta(\widetilde{q})^3$ and presents the y dependence of $f_{n,0}(y)$ as part of the extension. However, equation (3.18) of Ref. [27] already contains

   $$\eta\Big(\widetilde{q}\, e^{16\pi^2/\beta_{\text{JT}}}\Big)^3 .$$

   Using the convention of the present manuscript, $\beta_{\text{JT}} = 2\lambda\beta$, its argument is exactly

   $$\widetilde{q}, e^{8\pi^2/(\lambda\beta)} = e^{y/3}\widetilde{q} = \Lambda.$$

   Thus the full leading function $f_{n,0}(y) = -3\sigma_1(n)e^{ny/3}/n$, not only its $y = 0$ value, is already encoded in Ref. [27]. The manuscript partly acknowledges this near (5.38), but the abstract, the discussion around (1.1)–(1.2), and the conclusions do not consistently reflect it.

   The genuinely new claims appear to be the quasi-modular organization of the full large-x tower, the $O(\lambda)$ and $O(\lambda^2)$ corrections, and the exact differential equation together with its applications. These are worthwhile contributions. The paper should center its novelty statement on them and compare directly with both the low-energy equation (3.18) and the strict low-temperature equation (3.27) of Ref. [27].

5. **The asymptotic regime and the meaning of "exact" need to be stated consistently.**

   The perturbative discussion initially considers $\lambda \to 0$ at fixed β, whereas the nonperturbative reorganization (5.24) is explicitly a joint limit $\lambda \to 0$, $\beta \to \infty$ with $\lambda\beta$ (equivalently y) fixed. At fixed β, y grows as $1/\lambda$, so the series $\sum_p f_{n,p}(y)\lambda^p$ is not ordered in the same way. The introduction should not describe the result simply as a small-λ expansion at fixed β.

   The convergence domain of the Λ expansion should also be given. For positive real parameters, $|\Lambda| < 1$ requires $\beta > 2$; at $\beta = 2$ the proposed exponential suppression disappears. This is compatible with the intended low-temperature regime but should be stated. In addition, the interchange between the fixed-order Bessel asymptotic and the infinite sum over n is used formally. Either a uniformity argument should be provided or the formal asymptotic status should be made explicit.

   Finally, the exactness of equation (6.3) should not be conflated with uniqueness of the modular solution. The differential equation follows exactly from Bessel's equation, but it requires boundary data in q to determine a solution, and the divisor sums enter through those data, as the conclusions correctly note. The term "modular inversion" would also be more precise than "S-duality" unless a physical duality operation is intended and justified.

6. **Several displayed normalization errors must be corrected.**

   There are three clear inconsistencies in central formulas.

   - In equation (4.11), the anomalous term in the transformation of E_2 should be

     $$\frac{6}{\pi i}\, c(c\tau + d),$$

     not $c(c\tau + d)/(6\pi i)$. The printed coefficient is wrong by a factor of 36. Equation (4.14), and hence the subsequent specialized inversion formulas, use the correct normalization; the error nevertheless needs correction because (4.11) is the general formula on which the discussion is based.

   - For the large-x ansatz immediately preceding equation (6.5), the initial datum must be

     $$\phi_0 = S_0(q) = (q;q)_\infty^3,$$

     not $\phi_0 = 1$. With the printed value the recursion gives a q-independent first coefficient and cannot reproduce (4.22). If the authors instead wish to normalize ϕ_0 to one by factoring out S_0, the differential operator in the recursion must act on that factor and the displayed recursion changes. The following sentence, which identifies ϕ_k with the T_k sequence, already presupposes $\phi_0 = S_0$.

   - The ansatz for B_k in equation (A.14) is missing an overall factor of -3. At $k = 0$, (A.14) gives $B_0 = A_0$, contradicting the initial condition $B_0 = -3A_0$ in (A.5). The subsequent expression for $\Sigma^{(1)}$ restores the factor, so the intended ansatz is evidently

     $$B_k = -3A_0 \left(-\frac{144\pi^2}{\lambda^2}\right)^k \left(1 + B_k^{(1)}\lambda + B_k^{(2)}\lambda^2 + \cdots\right).$$

   - Equation (6.14) drops the factor 1/8 in the second term when (6.10) is evaluated at $p = 2\sin u$. The intermediate line should read

     $$f = \frac{2u}{\cos u}\, 2\sin u + \frac{1}{8}\left(\frac{2u}{\cos u}\right)^2 (4\sin^2 u - 4),$$

     which then gives the stated $f = -2u^2 + 4u\tan u$. Without the 1/8, the displayed intermediate expression is not equal to the displayed result.

   These appear repairable and do not by themselves invalidate the later formulas, but they obscure the logical derivation and should be corrected throughout.

Minor comments

1. The statement following equation (3.8) that the small-x series for $F(x)$ has a finite radius of convergence is incorrect for $0 \le q < 1$. The spectral integral has compact energy support and defines an entire function of x (equivalently of β), and the Bessel sum has the same property. It is the Taylor series for $\log F$ whose radius is limited by the nearest zero of F. The analyticity argument about the absence of hidden corrections near $x = 0$ can be retained after making this distinction.

2. The explanation in Section 5.3 that modular inversion maps each Eisenstein series to a prefactor times the same series overlooks the additive anomaly of E_2. The triangular-support conclusion can still be obtained because the anomaly generates lower-derivative terms multiplying S_0, but that closure should be shown explicitly. As written, the argument contradicts the correct transformation (4.14).

3. The phrase "exact in q" applied to (4.22) should always be accompanied by the qualification that the expansion is asymptotic in $1/x$. Conversely, when discussing the $e^{-4\pi^2/\lambda}$ sectors, the manuscript should distinguish nonperturbative effects in λ from the separate Stokes sector of the large-x expansion.

4. The identity $F(x)|_{q=1} = 1$ in (6.21) supplies useful boundary data. It would help to state whether $q = 1$ is taken as a convergent termwise limit of (2.7) or by continuation from $q < 1$, and to give the elementary Bessel identity used to establish it.

5. In the discussion immediately following equation (5.18), the expansion of the spectral edge is stated as $E_0 = 2/\lambda + 1/2 + \lambda/24 + \cdots$. The coefficient is $\lambda/48$, consistently with the expansion of $2/\sqrt{\lambda(1-e^{-\lambda})}$ used elsewhere in Section 5.

6. The operator $q\,d/dq$ is the logarithmic nome derivative. Calling it a "modular derivative" without qualification may cause confusion with the weight-covariant Serre derivative; a brief clarification would be useful.

7. The manuscript would benefit from a short ancillary file containing the symbolic generation of the coefficients in (4.22) and the checks leading to (5.35). This is particularly important because no numerical table is provided from which the stated check through $n = 20$ can be independently reconstructed.

8. The bibliography entry associated with the 2013 high-temperature combinatorial result contains visibly corrupted characters and should be repaired. Several minor typographical errors (for example, "in section" and inconsistent use of "non perturbative") should also be corrected in a final pass.

Recommendation

Major revision. The quasi-modular construction and exact differential equation are interesting and, after revision, should make the work suitable for a journal such as JHEP or Physical Review D. Publication requires an honest all-n status for (5.35), a restriction or completion of the saddle-spectrum claim, a correct comparison of Z-sector weights with the bulk saddles, and correction of the displayed normalization errors. These revisions would sharpen rather than diminish the paper's strongest contribution: a systematic organization of the subleading DSSYK low-temperature expansion beyond the previously known leading eta-function result.