Referee Report

Effective dynamics and quantum information in de Sitter wedge holography

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Recommendation: Reject in the present form. The subject is worthwhile, but several central derivations are incorrect and the claimed Page curve is not obtained from the equations displayed in the manuscript. A fundamentally rederived paper, with a more sharply isolated new result, could be considered as a new submission.

Summary and overall assessment

The manuscript studies a finite region of AdS_{d+1} written in a dS_d slicing and bounded by two end-of-the-world branes at $\rho = \rho_1$ and $\rho = \rho_2$. The proposed holographic description is a CFT_{d-1} associated with the codimension-two ends of the wedge. The authors reduce the bulk action along the radial direction, calculate sphere partition functions and complex extremal areas, study spin-two and vector Kaluza–Klein spectra, attempt to verify the first law of entanglement, construct a timelike pseudo-entropy saddle, and finally claim a Page-like transition between no-island and island surfaces.

There are useful ingredients in the analysis. In particular, the Sturm–Liouville form in equation (3.27) makes the qualitative Neumann zero-mode result transparent. With either Dirichlet or Neumann boundary conditions, integration by parts gives

$$m^2 = \frac{\displaystyle\int_{\rho_1}^{\rho_2} \sinh^d\!\rho\,(H')^2\,\mathrm{d}\rho}{\displaystyle\int_{\rho_1}^{\rho_2} \sinh^{d-2}\!\rho\,H^2\,\mathrm{d}\rho} \geq 0,$$

and the constant $m = 0$ mode is admitted by Neumann but not Dirichlet conditions. The radial norm appearing in equation (2.11) is also the expected zero-mode coefficient, and the smooth part of the Schrödinger potential has the standard warped-space form. The broad attempt to connect effective dynamics, complex dS/CFT observables, and information-theoretic quantities is potentially interesting.

Unfortunately, these sound qualitative observations do not rescue the main claims. The action reduction has a reversed boundary contribution and the subsequent dimension-by-dimension partition functions are mutually inconsistent. The Dirichlet determinant uses the degree and order of the associated Legendre functions in the opposite order from the stated radial solution. The first-law calculation changes the sign of the standard modular Hamiltonian and uses the real extremal-surface equation on a domain where the surface is complex. Most decisively, the purported higher-dimensional no-island profile is not an extremum of the area functional printed in the paper, while the island entropy is demonstrably time dependent. Thus Figure 11 is not a computed Page curve. These are foundational, rather than presentational, problems.

The novelty is also not yet sufficient for a paper at the level of JHEP or Physical Review D. The effective-action, spectrum, and first-law strategy substantially overlaps Ref. [20]; the dS-wedge bath and Page-curve setting is already central to Ref. [22]; and partition-function and extremal-area evidence for non-unitary codimension-two duals has a close antecedent in Ref. [18]. The potentially new directions here are the dS-specific massive spectrum, a fully defined timelike pseudo entropy, and an arbitrary-dimensional island phase diagram. Each of those directions is incomplete or technically incorrect in the present version.

Major comments

1. **The effective action and partition-function normalization must be rederived.**

   Equation (2.10) does not follow from equation (2.9) with the signs printed there. Define

   $$F(\rho) = \sinh^d\!\rho\, \coth\rho.$$

   The identity in equation (2.12) gives

   $$F(\rho_2) - F(\rho_1) = \int_{\rho_1}^{\rho_2} \Big[(d-1)\sinh^{d-2}\!\rho + d\sinh^d\!\rho\Big]\, \mathrm{d}\rho.$$

   At the normalization of equation (2.9), cancellation of the unwanted $\sinh^d\rho$ term and production of the cosmological term in equation (2.10) require the boundary contribution $+2[F(\rho_2)-F(\rho_1)]$. Equation (2.9) instead contains $+2[F(\rho_1)-F(\rho_2)]$. Holding the bulk integrand and the outward-normal conventions of Appendix A fixed, this is the opposite sign. The Gibbons–Hawking and tension terms, including their orientation on each brane, should be varied and evaluated again before any effective action is asserted.

   There is a second conceptual issue. Substituting a single ρ-independent metric $h_{ij}(x)$ and integrating over ρ isolates the common graviton zero mode. It does not derive the complete brane effective action for two independently fluctuating induced metrics. The latter contains the massive tower and, in a derivative expansion, higher-derivative or nonlocal terms of precisely the kind discussed in Ref. [20]. Equation (2.10) may be a useful zero-mode truncation after its signs are fixed, but it does not by itself establish an exact equivalence to pure Einstein gravity on both branes.

   The subsequent arithmetic is internally inconsistent with equation (2.16). Three examples are sufficient:

   - For $d=3$, equation (2.16) gives

     $$I_G = \frac{\cosh\rho_2 - \cosh\rho_1}{8G_N}\left(e^{2\tau_\infty} - e^{-2\tau_\infty} + 4\tau_\infty\right).$$

     Equation (2.18) has $1/(16G_N)$, whereas equation (2.20) silently returns to the $1/(8G_N)$ coefficient.

   - For $d=4$, let

     $$B = \sinh 2\rho_2 - \sinh 2\rho_1 - 2(\rho_2-\rho_1).$$

     Using $\omega_3 = 2\pi^2$ and $J_3 = e^{3T}/24 + 3e^T/8 + \cdots$ in equation (2.16) yields

     $$I_G = \frac{\pi B}{128 G_N} e^{3T} + \frac{9\pi B}{128 G_N} e^T + \cdots,$$

     not equation (2.23).

- Equation (2.25) contains e^{τ_∞}, while the exact J_4 and equation (2.26) require $e^{2\tau_\infty}$.

Finally, equations (2.6)–(2.8) formulate a Euclidean relation $Z = e^{-I}$, but equations (2.22) and (2.31) are extracted using $Z = e^{iI_G}$, the convention later stated in Appendix G. The paper needs one explicit continuation contour relating the Euclidean AdS wedge saddle, the Lorentzian dS wavefunction, and the defect generating functional. Until both the normalization and the phase are fixed, the reported imaginary anomaly coefficients are not a derived result, and their apparent agreement with Section 3 cannot be used as an independent check.

2. **The massive Dirichlet graviton spectrum is computed from the wrong special functions.**

The solution in equation (2.39) is

$$H(\rho) = \operatorname{csch}\rho \left[c_1 P_1^\lambda(\cosh\rho) + c_2 Q_1^\lambda(\cosh\rho)\right], \qquad \lambda = \sqrt{1-m^2},$$

where the degree is 1 and the order is λ. The determinant in equation (2.42) instead uses P_λ^1 and Q_λ^1. Degree and order are not interchangeable. The Dirichlet condition following from equation (2.39) is

$$P_1^\lambda(x_1)Q_1^\lambda(x_2) - P_1^\lambda(x_2)Q_1^\lambda(x_1) = 0, \qquad x_a = \cosh\rho_a.$$

Consequently, the massive Dirichlet roots and all figures based on equation (2.42) must be recomputed. The captions, prose, and plotted axes respectively use $\log|D+1|$, $\log|D|$, and $\log(|D|+1)$ (and similarly for N). These are three different diagnostics; only the last has a finite trough at zero when $D=0$. The definition used to generate each plot, together with the numerical root-finding procedure and residual tolerances, should be stated explicitly.

Appendix B compounds this problem by applying simple $\mu \mapsto -\mu$ identities for associated Legendre functions to the generally noninteger or complex order λ. For noninteger order, the continuation relates the independent P and Q solutions rather than multiplying each by the same $(-1)^\lambda$ factor used in equations (B.2)–(B.7). The claimed proof is invalid. Symmetry of the solution space can instead be discussed directly from the differential equation, which depends on λ^2.

The correct Sturm–Liouville argument does support a nonnegative discrete spectrum on a finite interval and a constant Neumann zero mode. The revised paper should build on that clean result: compute the corrected gaps, normalize the modes, determine their couplings to each brane, and check the massive spin-two states against the appropriate dS Higuchi/negative-norm criterion. A normalizable zero mode alone does not establish a healthy localized Einstein regime. The conformal-coordinate values in Figure 5 should also be corrected: $w = \log\tanh(\rho/2)$ gives $w(5) \simeq -0.0135$ and $w(2) \simeq -0.2723$, not the positive values quoted in the caption.

3. **The boundary theory and the observable called "entanglement entropy" are not defined consistently.**

The manuscript alternates among one CFT on one sphere, two timelike-separated defects at past and future infinity, and the factorization of UV- and IR-brane degrees of freedom in equation (4.1). These are different systems and different bipartitions. Moreover, the two constant-ρ branes do not intersect at finite bulk radius. The authors should specify precisely how many defect theories are present, where each lives, what state or transition amplitude relates the past and future components, and how that description maps to the UV/IR brane split used for the island calculation.

This distinction matters because the paper infers a non-unitary theory with imaginary anomaly coefficients and complex areas, yet repeatedly imports density matrices, positive relative entropy, and ordinary CFT entanglement formulas. In a non-unitary theory the relevant object may instead be a transition matrix and the geometric quantity a pseudo entropy. Section 5 acknowledges that this interpretation is unresolved; that acknowledgment is incompatible with the earlier claim of a verified ordinary entanglement first law. A revised manuscript should begin with a replica or wavefunctional definition of the observable and a complex-contour/homology prescription for the dS extremal surface. Only then can the coefficients in equations (3.3)–(3.6) be interpreted as a central-charge check. In any case, the partition-function and area coefficients share the same radial zero-mode norm, so agreement between them is not fully independent evidence for the proposed duality.

4. **The first-law calculation has a sign/domain error and does not establish the stated result.**

   Equation (3.21) gives the dS Poincaré extremal surface as

   $$x^2 - z^2 = \ell^2.$$

   For real z, its domain is $x \geq \ell$. Equations (3.24)–(3.25), however, integrate over $x^2 \leq \ell^2$, where $z^2 = x^2 - \ell^2 < 0$. Thus the computation has moved to a complex surface without specifying its contour, measure, or branch.

   There is also an immediate sign contradiction independent of that contour. The standard modular Hamiltonian written in equation (3.13) has positive kernel

   $$H_B = \frac{\pi}{\ell} \int_{x<\ell} (\ell^2 - x^2) T_{t_E t_E}.$$

   On the surface of equation (3.21), $\ell^2 - x^2 = -z^2$. Equation (3.38) should therefore contain $-z^2$, not $+z^2$. Equations (3.39)–(3.40) consequently produce the opposite of the modular Hamiltonian defined earlier. The negative modular-energy variation emphasized in Section 5 is generated by this sign change; it is not evidence for non-unitarity.

   Several further steps prevent the calculation from being repaired by a single sign correction:

   - Equation (3.45) retains only a leading Fefferman–Graham coefficient but uses it over an extremal surface reaching $|z| \sim \ell$. At nonzero momentum the full radial dependence is required. A controlled small-ball or long-wavelength expansion should be stated and kept consistently.
   - The passage from equation (3.51) to equation (3.52) effectively requires $(p^{t_E}/p^1)^2 = -1$. This is a special null analytic continuation, not a generic small-real-Euclidean-momentum limit. The entanglement first law must hold for every admissible first-order state perturbation.
   - Equations (3.57) and (3.59) omit the factor $\langle T_{t_E t_E} \rangle$ present in their defining integrals; as printed, the answers are not linear in the perturbation and have the wrong dimensions.
   - Direct substitution of equation (3.45) into equation (3.25) also produces the phase and perturbation factor inherited from $-i\epsilon$. Equation (3.47) drops these without supplying a contour factor that would cancel them.
   - Formulas involving Ω_{d-4} do not cover the $d = 3$ case emphasized in the spectrum analysis and require a separate treatment.

- Comparing powers in equations (3.42) and (3.43) is not a complete holographic renormalization argument that the massive modes make no stress-tensor contribution. Source terms, local terms, and the operator dual to the massive spin-two field should be identified.

The useful radial-orthogonality statement that massive modes have zero overlap with the constant mode can survive. The advertised first-law proof does not.

5. **The proposed no-island surface is not extremal for $d > 2$.**

Appendix D parameterizes the spatial sphere with transverse coordinates proportional to $\sin\theta$, thereby inducing $d\theta^2 + \sin^2\theta\, d\Omega_{d-2}^2$. Equations (4.6), (4.7), and (F.2) instead use $d\theta^2 + \cos^2\theta\, d\Omega_{d-2}^2$. The profile imported from Appendix D,

$$\tanh\rho_g\,\cos\theta = \cos\theta_0,$$

was therefore derived in the wrong angular convention for the area functional actually varied in Section 4.

This failure can be checked directly. Put $n = d - 2$ and

$$L = (\sinh\rho\,\cos\theta)^n\sqrt{(\rho')^2 + \sinh^2\rho}.$$

Substitution of the printed profile into $\frac{\mathrm{d}}{\mathrm{d}\theta}(\partial L/\partial\rho') - \partial L/\partial\rho$ leaves, up to the common positive normalization,

$$-\frac{n\,\cos^n\theta_0\,\cos^{n-1}\theta}{\sin\theta_0\,[\cos^2\theta - \cos^2\theta_0]^{n/2}},$$

which vanishes only when $n = 0$, i.e. $d = 2$. Hence equations (4.8)–(4.10), Appendix F, and the no-island curve in Figure 9 are not valid in the claimed $d > 2$ regime. The higher-dimensional extremal problem must be solved anew, with the correct region and homology class. Equation (A.5) should also be corrected: $\sinh\rho_g = \sinh\rho\cosh\tau$, not $\sinh\rho\cos\tau$.

6. **The island branch is not constant, so Figure 11 is not a Page curve.**

The boundary condition in equation (4.18) depends explicitly on τ_{UV}, and both the area functional (4.16) and the equation of motion (4.17) depend on $\tau(\rho)$. Computing one solution at $\tau_{\mathrm{UV}} = 2.7$ cannot justify the statement that $S_R^{(\mathrm{Island})} = 142.795$ is constant in time.

Indeed, the paper's own equations provide an exact counterexample. At $\tau_{\mathrm{UV}} = 0$, the profile $\tau(\rho) = 0$ solves equation (4.17) and both conditions in equation (4.18). For the same parameters used in the numerical example, $d = 3$, $\rho_1 = 1$, $\rho_2 = 3$, $\theta_0 = \pi/4$, and $G_N = 1$, equation (4.16) gives

$$S_{\mathrm{island}}(0) = \frac{\pi}{\sqrt{2}}\,(\cosh 3 - \cosh 1) \simeq 18.94,$$

whereas the manuscript reports $S_{\mathrm{island}}(2.7) = 142.795$. Thus the island entropy is strongly time dependent according to the displayed model. The horizontal red line in Figure 11 has no derivation.

Both saddles must be solved over a continuous range of the same boundary time, regulated in the same scheme, and compared after all divergent pieces and any generalized-entropy terms are included. The crossing and Page time should then be outputs rather than assumptions. The paper also first takes $\rho_2 \to \infty$ to decouple bath gravity but uses $\rho_2 = 3$ numerically without estimating the remaining gravitational coupling or renormalizing the corresponding UV divergence. Finally, the growth of the no-island expression is induced through a time-dependent cutoff/expanding dS region; it is not shown to be entropy carried by Hawking quanta. Because the background contains neither an evaporating black hole nor a specified radiation-producing state, even a corrected crossing would establish an extremal-surface transition, not restoration of microscopic unitarity. The caution in the final paragraphs of Section 5 should replace the stronger claims in Section 4.

7. **The timelike surface is not extremal in the full wedge and its numerical entropy is not reproducible.**

Even within the restricted constant-ρ ansatz, equation (3.62) has the wrong sign. For

$$L = (\cosh\tau\cos\theta)^{d-2}\sqrt{1-\cosh^2\tau\,(\theta')^2},$$

the Euler–Lagrange equation, divided by its nonzero common factor, is

$$\begin{aligned} 0 = \theta'' + d\tanh\tau\,\theta' - (d-2)\,\mathrm{sech}^2\tau\,\tan\theta + (d-2)\tan\theta\,(\theta')^2 \\ - (d-1)\cosh\tau\sinh\tau\,(\theta')^3. \end{aligned}$$

Equation (3.62) prints a plus sign in front of the $\mathrm{sech}^2\tau\tan\theta$ term. Figure 7 and the quoted area must therefore be recomputed.

There is a further, independent variational problem. Equation (3.61) fixes $\rho = \rho_0$ and varies only $\theta(\tau)$. There is no brane or other constraint at the arbitrary intermediate value ρ_0 that would forbid radial variations. The area density contains $\sinh^{d-1}\rho_0$; varying a constant-ρ embedding in the radial direction produces a term proportional to $(d-1)\coth\rho_0$, which is nonzero at finite ρ_0. The displayed solution therefore does not solve the complete codimension-two extremality equations.

Moreover, the value $1.3396\,i$ is quoted without the finite value of τ_0 used in the numerical boundary conditions, a cutoff subtraction, convergence data, or an evaluation of the disconnected competitor. For fixed nonzero $\cos\theta_0$, the raw integrand grows at large $|\tau|$, so the limit $\tau_0 \to \infty$ requires a definite renormalization prescription. The factor of two attributed to a $\mathbb{Z}_2$ symmetry and the assumed dominance for nearby branes also need derivation for generic ρ_1, ρ_2. A publishable treatment should vary both ρ and θ, specify the complex contour and homology condition, renormalize the area, and compare all saddles.

8. **The paper must isolate a genuinely new result relative to the existing literature.**

Ref. [20] already develops the wedge effective action, the Neumann massless/Dirichlet non-massless graviton conclusion, and the radial-orthogonality mechanism by which massive modes drop out of the first-order entropy. Ref. [22] introduces the AdS dS-wedge setting with UV-brane bath, islands, and a Page-like transition. Ref. [18] gives closely related codimension-two duals with dS boundaries and uses partition functions and holographic areas to infer non-unitarity. The present manuscript should compare formulas and assumptions with these works result by result, rather than treating the collection as independent evidence for a new dictionary.

There are plausible routes to a substantial contribution: a corrected dS spin-two spectrum including Higuchi bounds and brane couplings; a fully defined dS pseudo-entropy and first law; or a genuine higher-dimensional phase diagram obtained from correct extremal surfaces. At present, however, the central qualitative conclusions are inherited, while the claimed new quantitative extensions are the parts that fail.

Minor and presentational comments

1. The normalization of equation (2.1) is incompatible with a standard Einstein–Hilbert plus Gibbons–Hawking action even after setting $16\pi G_N = 1$: the boundary term must retain the relative factor of two. The restored factors in equation (2.13) use a different convention.

2. The induced metric on a brane is $\widehat{h}_{ij} = \sinh^2 \rho_a \, h_{ij}$. The manuscript frequently uses the same symbol h_{ij} for the unit-radius dS metric and the physical induced metric. This obscures the tension relation, the outward-normal signs, and the meaning of the off-shell variation.

3. The figures displaying spectral functions should list the actual roots, their numerical precision, and the branch convention for P_ν^μ and Q_ν^μ when λ is imaginary. Root markers alone are not adequate for a reproducible KK analysis.

4. In Appendix C, $n = 0$ is listed among the formal Dirichlet roots in equation (C.12), although the separately solved $m_v = 0$ equation admits only the trivial Dirichlet solution. The physical Dirichlet tower should begin at $n = 1$.

5. Equation (4.17) is cross-referenced as equation (4.2) because its label is placed outside the equation environment. This should be corrected along with the unmatched closing delimiter in equation (4.16).

6. In the discussion around equation (4.5), ρ_g, not r, is the global AdS radial coordinate. The coordinate transformation should be stated once consistently and checked against the dS slicing throughout Section 4.

7. The numerical inputs underlying Figures 7, 9, 10, and 11 should include cutoff values, solver tolerances, residuals of the Euler–Lagrange equations, and convergence under grid refinement. The current plots cannot be independently checked from the text.

8. The manuscript would benefit from a sharper separation between ordinary entanglement entropy, pseudo entropy, generalized entropy, and a complex extremal area. These terms are currently used too interchangeably for a paper whose main physical conclusion depends on precisely that distinction.

Recommendation

The questions addressed by the manuscript are timely, and the finite dS-sliced wedge is a potentially useful laboratory. Nevertheless, the correctness, novelty, and significance standards for publication are not met in the present form. The wrong action sign, inconsistent partition coefficients, incorrect Dirichlet determinant, failed first-law proof, non-extremal higher-dimensional no-island profile, and demonstrably nonconstant island branch affect essentially every headline result. I therefore recommend rejection. A substantially reconstructed paper focused on one corrected and genuinely new dS-specific result could merit fresh consideration.