

Referee Report

Group Theory and the CFT Distance Conjecture:

$\mathcal{N} = 2$ Tensionless Strings Have No (Co)Weight

arXiv:2607.12014

Recommendation: Major revision. The manuscript could become suitable for publication after the central large- N limits, Gaussian formulas, distance normalization, and holographic claims are corrected and more sharply delimited.

Summary and overall assessment

This manuscript studies the thermal partition function at overall weak coupling for four-dimensional large- N Lagrangian $\mathcal{N} = 2$ superconformal gauge theories. Its organizing observation is that the matter adjacency of a quiver can be encoded by a symmetrizable Cartan matrix. Quivers with only bifundamental matter are summarized by data $(\mathfrak{b}, S^i, f^i)$, while mixed unitary/orthogonal/symplectic quivers and theories with two-index matter are related to simply-laced diagrams by unfolding or doubling. The authors then evaluate the large- N gauge-singlet partition function as a Gaussian integral over power sums. They argue that the first thermal singularity is controlled by the spectral radius of the relevant Cartan adjacency. For affine diagrams this radius is 2; for finite diagrams it is $2 \cos(\pi/h)$, or equivalently $2 \cos(\pi/h^\vee)$ after unfolding to ADE type.

The manuscript uses this result to define classes with a common leading Hagedorn exponent. Affine quivers are related to orbifold and orientifold daughters of $\mathcal{N} = 4$ super-Yang–Mills, whereas most finite quivers are obtained by deleting a node from an affine diagram. The authors discuss brane engineering, possible six-dimensional SCFT/Little String Theory origins, and a proposed F-theory double-scaling limit. Finally, they relate the same Cartan spectral data to the weak-coupling Zamolodchikov metric and derive bounds on decay exponents in partial-decoupling limits.

There is a valuable paper in this framework. The group-theoretic synthesis is elegant, the extension beyond purely unitary quivers is potentially useful, and the cosine formula gives a memorable organization of a large collection of free-field counting problems. The connection to the one-node examples also provides a helpful conceptual explanation of the three values $\lambda_{\max} = 2, 0, 1$ in the $\mathcal{N} = 2$ subset. These results should interest readers working on large- N gauge theory, conformal manifolds, and tensionless limits.

In its present form, however, the manuscript does not yet establish all of its main claims. There are explicit sign and normalization errors in the Gaussian partition functions; the strict large- N treatment of every finite-base family is complicated by flavour ranks of order N ; the exponent called κ is defined inconsistently between the Introduction and Section 5; and the proposed double-scaling limit in equation (4.39) is algebraically incompatible with the preceding volume relations. Several conclusions also promote necessary central-charge tests or suggestive Hagedorn evidence into claims of controlled holography or identified bulk strings. I therefore recommend a substantial revision rather than acceptance in the current version.

Novelty and significance

The genuinely new part appears to be the uniform Lie-spectral treatment of the free large- N thermal kernel across $\mathcal{N} = 2$ quivers with all classical gauge algebras, including folded diagrams and many cases with symmetric or antisymmetric matter. The extension of the distance-rate discussion from unitary quivers to this broader set is also potentially new. By contrast, McKay engineering of affine quivers, orientifold constructions, affine-node deletion, six-dimensional SCFT/LST tensor decoupling, the proposed little-string description of superconformal QCD, and the original one-node universality classes are developments from the cited literature. Since equation (3.36) depends only on the Coxeter number of the unfolded graph, distinct ADE diagrams with the same Coxeter number also share a temperature; the actual temperature classes are therefore Coxeter-number classes, with ADE diagrams providing their realizations.

The revision should make this division explicit. In particular, the comparison with the recent unitary-quiver analysis and with the simultaneously submitted brane-model work should state precisely which formulas and families are new here. Statements such as “the same string-like spectrum” should be replaced by the result actually shown: equality of a leading Hagedorn temperature in a specified free, large- N limit. Equality of this exponent does not imply equality of the full spectrum, polynomial prefactors, twisted/open sectors, or the identity of the bulk string.

Major comments

1. **The distance exponent is defined with a factor-of-two inconsistency, and the AdS mass/dimension statement should be corrected.**

Equation (1.1) defines the anomalous dimension by

$$\gamma_{\text{HS}} \sim e^{-\kappa \text{dist}}.$$

Equations (5.3)–(5.4), however, define $g_i \sim e^{-\kappa_i \hat{g}_i}$ and then obtain

$$\gamma_i \sim g_i^2 \sim e^{-2\kappa_i \hat{g}_i}, \quad M_{\text{HS}}/M_{\text{Pl}} \sim \sqrt{\gamma_i}.$$

Consequently, the κ_i appearing in equations (5.3), (5.6), and (5.9), including the bound $1/\sqrt{2}$, is the exponent governing the mass (or $\sqrt{\gamma}$), not the exponent governing γ as equation (1.1) presently states. If the authors retain the Section 5 convention, equation (1.1) should read $\sqrt{\gamma_{\text{HS}}} \sim e^{-\kappa \text{dist}}$, equivalently $\gamma_{\text{HS}} \sim e^{-2\kappa \text{dist}}$. If instead κ is to denote the anomalous-dimension exponent, all quoted bounds must be rescaled by two.

The nearby Introduction also says $mL \propto \Delta - \Delta_{\text{bound}}$. Near a higher-spin conservation point, the standard AdS relation gives $m^2 L^2 \propto \Delta - \Delta_{\text{bound}}$, and hence $mL \propto \sqrt{\gamma}$, consistently with equation (5.4). This normalization issue propagates through the abstract, equations (1.1) and (1.5), the interpretation of equation (5.9), and the conclusions. It must be repaired globally before comparison with the CFT Distance Conjecture is meaningful.

2. **The Cartan classification argument in Section 2 uses an incorrect determinant criterion and ambiguous representation-theory normalization.**

The paragraph surrounding equations (2.7)–(2.9) states that a generalized Cartan matrix is finite when $\det C > 0$, affine when $\det C = 0$, and indefinite otherwise. Determinant sign

alone is not the classification criterion in rank greater than two: finite type requires positive definiteness (equivalently positivity of the relevant principal minors), while affine type requires positive semidefiniteness of corank one with the appropriate indecomposability conditions. An indecomposable indefinite symmetrizable Cartan matrix can have positive determinant if it has an even number of negative eigenvalues.

The desired result can likely be proved cleanly using the nonnegative adjacency $A = 2\mathbf{1} - C$ and Perron–Frobenius/Collatz–Wielandt arguments. The large- N SCFT condition and positivity of the leading rank/flavour vectors should imply $\rho(A) \leq 2$, with equality in the affine case and strict inequality in the finite case. The manuscript should supply this argument, state connectedness assumptions, and distinguish a solution of the one-loop beta-function equations from a fully consistent Lagrangian SCFT. This distinction matters for the exceptional candidates that are later called SCFTs while their global consistency is left open.

Equation (2.15) also appears inconsistent with the manuscript’s own symmetrization. Since $C = D^{-1}G$ and G is symmetric, $\theta^T C = 0$ implies $GD^{-1}\theta = 0$ and hence

$$C(D^{-1}\theta) = 0.$$

Thus the right null vector is proportional to $D^{-1}\theta$, up to integral normalization, rather than $D\theta$ as printed. The same relation is used in the subsequent affine rank scaling. The authors should reconcile marks, comarks, the placement of D , and their index conventions using at least one non-simply-laced affine example, and then propagate the corrected null vector through Sections 3–5.

There is also a systematic terminology error in equation (2.3) and Table 1. The quantities assigned the values $1/2$ for an $SU(k)$ fundamental and $h^\vee$ for the adjoint are Dynkin indices $T(R)$ in the stated normalization, not quadratic Casimirs $C_2(R)$. The two are related by $C_2(R) \dim R = T(R) \dim \mathfrak{g}$. The beta-function formula and the representation-exhaustion discussion should be rewritten with a consistent convention.

3. Several displayed Gaussian formulas have sign or factor errors even before the unfolding step.

Equations (3.10)–(3.11) imply, for a real SO/USp power-sum mode, that the Haar measure plus the vector multiplet contributes

$$-\frac{1 - z_V(x, n)}{2n} (p_n^2 + Sp_{2n}).$$

Accordingly, in the bracket in equation (3.17), the p_{2n} term should have coefficient $+(S/2)(1 - z_V)$, rather than the displayed $-S(1 - z_V)$. Completing the square then gives, with the conventions of the manuscript,

$$J_i(x, n) = z_H(x, n) \chi_{f^i}(V^n) - \delta_{2|n} S^i [1 - z_V(x, n/2)],$$

not the plus sign and factor 2 in equation (3.21). This corrected coefficient also agrees with the $\mathcal{N} = 4$ result in equation (3.13). The constant vector contribution in equation (3.17) has the wrong sign as well: for a unitary node, $\chi_{\text{adj}} = |p_n|^2 - 1$ produces $-z_V/n$, as equation (3.18) correctly reflects.

Similarly, the symmetric/antisymmetric matter term in equation (3.25) enters the plethystic exponential positively. With the overall minus sign used in that equation, the displayed $z_H[(x_n^1)^2 -$

$(y_n^1)^2 \pm x_{2n}^1]$ term should therefore carry the opposite sign. Swapping M_+ and M_- may leave the determinant product unchanged, but the numerator and source terms do not remain unchanged. The authors should rederive equations (3.17)–(3.26), show the completed squares explicitly, and verify them against simple SU , SO , and USp one-node examples.

4. The central unfolding/doubling derivation is too schematic to support the claimed all-classical result.

Equation (3.23) jumps, with ellipses, from a mixed real/complex Gaussian to the determinant of the adjacency of an unfolded ADE diagram. Since this is the principal extension beyond the previously studied unitary families, the explicit linear transformation is essential. The paper should display the real block matrix, the change from p_n to q_n , its Jacobian, and the treatment of linear flavour sources. A table covering every folded and twisted family would make clear which $\mathfrak{b}$ maps to which $\mathcal{U}(\mathfrak{b})$ and which gauge-node multiplicities are used.

The same applies to equations (3.27)–(3.28). Rather than saying the determinant identity follows “by inspection,” the authors should exhibit the similarity transformation that block-diagonalizes the doubled adjacency into the A_+ and A_- sectors, and state the allowed placements and multiplicities of two-index matter. The isolated SO two-index case treated only by reference should be identified as imported input rather than as part of the new derivation.

One local statement should also be removed: after equation (3.19), the manuscript says that an affine Cartan matrix is singular but its adjacency A is invertible. This is false; for example, affine $A_3^{(1)}$ has zero adjacency eigenvalues. Only $M(x, n)$ needs to be generically invertible, which is true away from its critical temperatures.

5. For finite diagrams, the strict large- N flavour scaling may invalidate the confined Gaussian saddle used to identify a physical Hagedorn temperature.

This is a central scope issue. For a finite invertible C , taking every gauge rank as $k_i \sim N$ in equation (2.16) necessarily makes at least one flavour rank f^i of order N . Thus the caveat after equation (3.38) is not merely a possibility for finite quivers; it is inherent in the large- N families under consideration. At unrefined fugacity $V = 1$, the flavour character in equations (3.18) and (3.20) is then of order N . The Gaussian source moves the saddle to $p_n = O(N)$, where positivity constraints on the eigenvalue density and possible flavour-driven saddles cannot be discarded. The statement that the strict large- N partition function is divergent at every temperature is also in tension with treating its later determinant zero as an ordinary thermal Hagedorn transition.

The authors should specify the observable and order of limits. Possibilities include taking the high-energy limit at each finite N before a Veneziano limit, keeping flavour fugacities generic and defining an appropriate coefficientwise limit, projecting to the flavour-singlet sector, or normalizing/subtracting the extensive flavour contribution. Whichever choice is intended, it must be shown that the uniform-density Gaussian phase persists up to the root in equation (3.33) and is not pre-empted by another saddle. Until this is done, the claimed physical universality of all finite-base Hagedorn temperatures is formal rather than established.

The location of the first determinant singularity also deserves a proof. Because $z_V(x, n)$ and $z_H(x, n)$ depend on the parity of n , the manuscript should show that no $n > 1$ factor vanishes before the $n = 1$ Perron eigenvalue. The essential singularity discussed in equations (3.37)–(3.38) should be treated with the correct approach-side sign: it is of the form $\exp[A/(x_H - x)]$, not $\exp[1/(x - x_H)]$. Its inverse-Laplace enhancement is $\exp[2\sqrt{AE}]$ with a coefficient set by the source projection; it is not universal. Finally, since $\lambda_{\max} = 2$ produces the earliest root, the

affine/ $\mathcal{N} = 4$ value is a *lower* bound on T_H (and an upper bound on the inverse temperature or exponential growth rate), contrary to the wording near the end of Section 4.4.

6. The F-theory double-scaling limit in equation (4.39) is reversed relative to the preceding equations.

Let $V_i = \text{Vol}(\Sigma^i)$ and $V_{\text{LST}} = \sum_i \ell_i V_i$, as in equation (4.36). The ratios $r_i = V_i/V_{\text{LST}}$ obey the identity

$$\sum_i \ell_i r_i = 1.$$

Equation (4.39) sends $r_0 \rightarrow 0$ while holding the other ratios fixed. At least one $r_{i \neq 0}$ must then be nonzero. Since the preceding paragraph sends $V_{\text{LST}} \rightarrow \infty$ to decouple the little string, that V_i also tends to infinity, and equation (4.34) sends another four-dimensional gauge coupling to zero instead of keeping it finite.

For the explicit SQCD limit in equation (4.37), with the torus area held fixed, $g_0 \rightarrow 0$ and g_1 fixed imply $V_0 \rightarrow \infty$ and V_1 fixed. Hence one obtains

$$\frac{V_0}{V_{\text{LST}}} \rightarrow \frac{1}{\ell_0}, \quad \frac{V_{i \neq 0}}{V_{\text{LST}}} \rightarrow 0,$$

which is the reverse of equation (4.39). If a simultaneous scaling of the torus area is intended, it must be stated and shown to reproduce all desired four-dimensional couplings. The claimed universal double-scaling construction and the conclusions drawn from it need to be rederived. The text should also distinguish the little-string tension $T_{\text{LST}} \propto V_{\text{LST}}$ from a mass scale, which normally scales as $\sqrt{T_{\text{LST}}}$.

More broadly, the universal six-dimensional lift is not established for the full claimed class. The footnote after equation (4.33) itself notes that six-dimensional anomaly cancellation gives $Ck = f - 16S$, whereas the four-dimensional condition is $Ck = f - 4S$ for orthogonal/symplectic cases, and then merely asserts a flow to the desired four-dimensional SCFT. The revision should either restrict the geometric construction to the unitary ADE families actually demonstrated, or provide the matter-changing mechanism and anomaly-consistent lift for every additional family.

7. Section 5 should formulate ray-dependent partial decoupling before quoting universal rates.

Equations (5.2)–(5.3) introduce orthonormal weak-coupling coordinates. A unit-speed ray has $\hat{g}_i = v_i d$ with $\sum_i v_i^2 = 1$, so the i th mass exponent along that ray is $\kappa_i v_i$, not simply κ_i . If a subset S of nodes is required to become weak at a common leading rate, the standard optimization gives

$$\alpha_{\min}(S) = \left(\sum_{i \in S} \kappa_i^{-2} \right)^{-1/2} = \sqrt{\frac{2c}{\sum_{i \in S} \dim \mathfrak{g}_i}}.$$

The $\kappa_{\min}$ of equation (5.6) is the case in which all nodes participate. Coordinate-axis rates κ_i , the optimized leading rate for a subset, and the overall-free rate are different quantities and should not share ambiguous prose.

The authors already cite the taxonomy/convex-hull literature, so this structure can be incorporated without changing the purpose of the paper. It would also clarify which higher-spin currents are being followed. Bifundamental fields on the boundary of a decoupled subquiver remain coupled to non-decoupled nodes, and their anomalous dimensions need not be governed by a

single g_i^2 . The manuscript should identify the tower whose mass realizes each quoted exponent and state whether its Hagedorn count includes the free vector, the interacting remainder, or a flavour-singlet projection.

8. The explicit affine-rate formula and the upper-bound claims are not valid with the stated scope.

For mixed unitary/orthogonal/symplectic affine quivers, equations (4.24) and (5.6) give

$$\kappa_i^2 = \frac{K \cdot D \cdot K}{2D_{ii}K_i^2}$$

at leading large N . Equation (5.10), $\kappa_i^2 = (K \cdot K)/(2K_i^2)$, follows only when D is proportional to the identity. This missing D dependence matters precisely for the mixed non-simply-laced families advertised as a new part of the paper.

Equation (5.11) bounds $\kappa_{\min}$, not arbitrary node rates κ_i , which remain unbounded as the manuscript itself observes. Its sharper upper bound is controlled by $R_b^{\max}$ or $h_b^\vee$, not solely by the $\lambda_{\max}$ that sets the Hagedorn root. The abstract’s claim that both lower and upper bounds are set by the same eigenvalue should therefore be weakened. The final strict inequality in equation (5.11) also fails for A_1 : Table 5 gives $R_{A_1}^{\max} = 2 = 4/h^\vee$, so $\kappa_{\min} = \sqrt{2/3}$ and there is no strict improvement over the quoted collider value in that case. The premise $\mu^i \geq 2$ used below equation (4.30) is likewise false for unflavoured internal nodes; nonnegativity is the relevant condition and the entrywise argument should be restated accordingly.

The statement that affine theories saturate equation (5.9) “at all values of N ” should also be qualified. Equation (4.25) establishes $n_H/n_V = 1 + O(1/N)$ for a general affine quiver. Exact finite- N saturation holds in special cases such as an $\mathcal{N} = 4$ vector plus one adjoint full hypermultiplet, not automatically for every flavoured orientifold daughter.

9. The holographic and string interpretations should be separated from the demonstrated spectral statements.

The inference around equations (4.21)–(4.25) is too strong. The condition $a/c \rightarrow 1$ is necessary for an Einstein-like bulk, but it is not sufficient; one also needs a suitable strong-coupling region, a large gap, and an appropriately sparse single-trace spectrum. At the overall-free point the higher-spin currents explicitly preclude an Einstein description. The revision should distinguish: (i) known top-down string backgrounds for fixed orbifold/orientifold data at large N and large ’t Hooft coupling; (ii) the necessary central-charge test; and (iii) the tensionless free cusp.

Relatedly, the free planar limit is not characterized simply by $g_s \rightarrow 0$. Since $g_s \sim \lambda/N$, it already tends to zero at fixed ’t Hooft coupling as $N \rightarrow \infty$; the string tension in AdS units is controlled by $L^2/\alpha' \sim \sqrt{\lambda}$ and becomes small when $\lambda \rightarrow 0$. The geometric quotient also depends on the orientifold: an $O3$ involution leads to an $\mathbb{R}P^5$ -type quotient (with discrete data), whereas an $O7$ involution has a fixed locus and D7/open-string sectors and is not generically described by $\mathbb{R}P^5/\Gamma$. The opening of Section 4.4 should not group these cases into one background.

For finite diagrams at fixed quiver rank, the central-charge ratio fails the Einstein test; in a second limit with the number of nodes large, the ratio may approach one but extra species and the order of the N , quiver-rank, and energy limits become important. Hagedorn growth alone does not identify a fundamental bulk string. The LST compactification and SQCD worldsheet discussion are useful motivation, but the generalization to all finite quivers should be labeled

as a proposal and accompanied by the evidence that would test it. The conclusion should also retain the three exceptional candidates and the unverified seven $\mathcal{N} = 1$ one-node theories instead of claiming that all finite theories and all members of the mini-landscape have been obtained from the proposed parent construction.

Minor comments

1. The half/full-hypermultiplet terminology should be standardized. Section 2 first correctly associates half-hypermultiplets with pseudoreal representations, but later calls them real. In $\mathcal{N} = 2$ language, $\mathcal{N} = 4$ super-Yang–Mills contains one full adjoint hypermultiplet, equivalently an $SU(2)_L$ doublet of two half-hypermultiplet letters in the convention of equation (3.4), not two full $\mathcal{N} = 2$ hypermultiplets. The $O7/D7$ example should similarly say one two-index antisymmetric hypermultiplet and eight fundamental half-hypermultiplets (or the equivalent full-hyper count).
2. The paragraph preceding equation (3.24) refers to an outlier with antisymmetric matter for $SO(k)$, whereas equation (2.19) displays the relevant $USp(k)$ case. For $SO(k)$, Λ^2 is the adjoint. Correct the group and state whether that outlier is derived here or imported from the earlier one-node calculation.
3. The $SO(2N + 1)$ shift $p_n^{SO(2N+1)} = p_n^{SO(2N)} + 1$ is harmless for the determinant but generates linear and constant terms in the numerator. Those terms matter for the essential singularity. Either include them in the claimed full partition function or restrict the detailed numerator formulas to even orthogonal groups.
4. In Section 4.1 the affine node is a conventional distinguished node whose deletion gives the standard finite diagram; it is not the unique gauge node whose exactly marginal coupling can be sent to zero. Other node deletions are physical and can yield different or disconnected subdiagrams. The claim that a proper decoupling never preserves $\lambda_{\max} = 2$ should be justified by Perron–Frobenius/interlacing for proper principal subgraphs, not merely by saying that a null eigenvalue is lost.
5. The prose in Section 4.1 alternately says that decoupling creates new short/BPS states through recombination and that the temperature changes because those states are absent. Please identify the two Hilbert spaces being compared and say precisely which states appear or disappear.
6. Equation (4.9) gives $Ad = 3d$ for the positive vector $d_i = \dim R_i$, but the thermal kernel is said to involve $(A + A^T)/2$. To conclude that its spectral radius is 3, show also $A^T d = 3d$ (from the conjugate representation/anomaly condition) and apply Perron–Frobenius with the needed connectedness assumptions.
7. The $\mathcal{N} = 1$ mini-landscape discussion establishes the parent/decoupling picture explicitly only for the eleven $\mathcal{N} = 2$ theories. For the remaining seven, provide a self-contained table of matter content, thermal adjacency, proposed parent, and decoupled node, or restrict the claimed explanation to the $\mathcal{N} = 2$ subset. A citation to a simultaneous unpublished analysis is not a substitute for the argument made in this manuscript.

8. In equation (4.21), n_H and n_V are dimension-weighted effective multiplet counts, not literal numbers of multiplet species. Define this at first use. Around equation (4.24), the relevant McKay identity is $\sum_i K_i^2 = |\Gamma|$, not that the sum of the comarks equals $|\Gamma|$.
9. Replace “finitely many twisted modes at a given excitation level” with “a finite number of twisted sectors.” Each twisted sector contains an infinite tower, even though a fixed finite projection does not change the leading exponential slope.
10. Appendix A should call $k_i \alpha^{i_V}$ with nonnegative coefficients an element of the positive coroot-lattice cone/monoid. Such a linear combination need not itself be a coroot. Likewise, “indefinite type” is the standard term for the generalized Cartan classification.
11. Replace “rank-one theory” by “one-node quiver” where the argument concerns the number of gauge factors. Explain explicitly why the folded diagram and its unfolded ADE diagram have the same spectral radius/Coxeter number in every family used in equations (3.36), (4.28), and (5.9).
12. The conifold has the same bifundamental sector as the stated orbifold, not the same complete matter content, since the missing adjoint chirals are precisely the difference noted in the next sentence. The anomalous-dimension convention in equation (4.17) and the restriction to the node-exchange-symmetric locus of the Klebanov–Witten conformal manifold should also be stated.

Recommendation

The Lie-algebraic organization of the thermal determinant is original enough to merit serious consideration, and the central cosine formula is likely to remain useful after revision. I would support publication if the authors (i) correct the Gaussian and distance-normalization errors, (ii) give a controlled definition of the finite-quiver large- N Hagedorn observable, (iii) supply the missing unfolding and first-singularity arguments, (iv) rederive or withdraw equation (4.39), and (v) sharply separate proven free-field spectral results from conjectural bulk interpretations. These changes are substantial, so I recommend major revision rather than a minor editorial round.