Referee Report

Strong-to-Weak Spontaneous Symmetry Breaking from Wormholes in Holography

Manuscript: arXiv:2607.12022
Recommendation: **Major revision**

Summary and overall assessment

The manuscript studies strong-to-weak spontaneous symmetry breaking (SWSSB) in fixed-charge thermal states and proposes a holographic realization in $\mathrm{AdS}_3/\mathrm{CFT}_2$. The field-theory diagnostic is a long-distance Rényi-2 or Wightman four-point function in a density matrix with a strong U(1) symmetry, while ordinary charged two-point functions remain short-ranged. The authors first recapitulate a clustering and ensemble-equivalence argument for lattice systems. They then consider Einstein gravity coupled to U(1) Chern–Simons fields and a charged probe scalar. A fixed-charge projector is represented by an integral over an imaginary chemical potential, the scalar Green function is computed in Euclidean BTZ, and large-N factorization is used to claim the nonzero bounds in equation (18), with details in equations (S34) and (S37). The final sections interpret the factorized channel as a wormhole, reformulate weak symmetry as an ensemble-emergent symmetry, and propose a swampland SWSSB conjecture.

There is a worthwhile idea here. In particular, applying symmetry-resolved thermal techniques to a nonlinear mixed-state order parameter in holography is interesting, and the combination of the charge-projector integral with the BTZ probe calculation could provide a useful explicit example. The qualitative mechanism is also plausible in an interior finite-density phase: if the relevant neutral composites cluster, both local thermal factors remain nonzero, and local fixed-charge/grand-canonical equivalence holds, the factorized channel indeed produces long-range Rényi order.

In its present form, however, the paper does not meet the standard for publication in a journal such as JHEP or Physical Review D. The lattice conjecture is false as stated, the argument supporting it changes temperature without justification and proves positivity of only one of two required factors, and the supplemental BTZ calculation contains several exact inconsistencies. Most importantly, the radial solution has the wrong boundary indices, the holographic boundary flux misses a temperature-dependent factor, and the numerical prefactors and Fourier projection in equation (S34) do not follow from the preceding equations. The claimed quantitative bound in equation (18) therefore is not established. In addition, the wormhole, ensemble-average, and swampland interpretations go substantially beyond what the calculation demonstrates. These problems require a substantive recalculation and a significant narrowing or independent justification of the conceptual claims.

Major comments

1. **The Lattice SWSSB Conjecture is false without essential hypotheses.**

   Consider hard-core bosons, or equivalently a spin-1/2 lattice, with $Q = \sum_i n_i$ and a local U(1)-invariant Hamiltonian such as $H = \Delta Q$. In the fixed sector $Q = 0$, equation (7) gives the product vacuum

   $$\rho_{0,\beta} = |0\cdots0\rangle\langle0\cdots0|$$

   for every finite β. This state has no conventional charged long-range order. Nevertheless, for any separated single-site charged operator, one of the raising/lowering operations annihilates the extremal state, so both the Rényi-2 and Wightman four-point diagnostics vanish. The maximal-filling sector gives the analogous counterexample. Thus locality, U(1) symmetry, finite temperature, and absence of ordinary SSB do not imply SWSSB.

The conjecture would need, at minimum, an extensive charge density in the interior of its allowed range, nonzero entropy density, a finite matching chemical potential, locally faithful reduced thermal states, and a charged operator having nonzero spectral weight in both required directions. These conditions should be stated precisely, and the claim should be presented as a conditional mechanism rather than a general conjecture unless the authors can formulate and support a corrected theorem.

There are three further gaps in the proposed argument. First, the hypothesis concerns absence of SSB in $\rho_{Q,\beta}$, whereas equations (10)–(13) for the Rényi-2 correlator use $\rho_{Q,2\beta}$. Disorder at temperature $1/\beta$ does not imply disorder at $1/(2\beta)$; a U(1) model in three or more dimensions can cross an ordering transition between these temperatures. Second, equation (11) contains the two distinct local factors

$$\langle \mathcal{O}(-i\beta)\mathcal{O}^\dagger \rangle_{Q,2\beta}, \qquad \langle \mathcal{O}^\dagger(-i\beta)\mathcal{O} \rangle_{Q,2\beta}.$$

Equations (12)–(13) discuss only the first. The second can vanish, as it does in the extremal-sector example, and both need a volume-independent positive lower bound. Third, absence of a charged two-point order parameter does not by itself imply clustering of the neutral composites used in equation (11). Extremality of the Gibbs state and neutral-sector clustering must be separate hypotheses.

2. **The definition and thermodynamic limits of the order parameter must be made precise.**

   Equation (5) writes the large-distance correlator as $R^{(2)}(x,y) = O(1)$. Big-$O(1)$ includes an identically zero function and therefore does not express long-range order. The criterion should require a strictly nonzero limit or liminf, with a specified normalization. The order of limits is also essential: one needs a sequence $L \to \infty$, a specified scaling of $Q(L)$, and then a large-separation limit within the infinite-volume state. The roles of c or N, L, and $|x-y|$ should be stated alongside the principal claim.

   This is not merely formal. In the Fourier series displayed in equation (S34), the behavior differs sharply between fixed total Q and fixed nonzero density Q/L. At fixed density, the hyperbolic-cosine factor competes nonuniformly with the Gaussian and with $f_\ell \sim \ell^{-2}$; a termwise $L \to \infty$ limit is not justified. A lower bound by a positive constant also does not show that the correlator is finite and volume-independent. The authors should control the full projected expression in the intended thermodynamic scaling, not only retain its $\ell = 0$ term.

   The holographic calculation establishes decay of the ordinary two-point function only for the chosen light probe with $\Delta_+ = 2$. Yet the definition surrounding equations (5)–(6) asks that conventional long-range order be absent for charged order parameters generally. Either a general finite-temperature CFT argument and the absence of charged bulk instabilities should be supplied, or the result should explicitly be described as an operator-specific Rényi long-range-order signal.

3. **The thermodynamic continuation and the charge/holonomy conventions are inconsistent.**

   Equation (S6) has the imaginary-chemical-potential Gaussian

$$\log Z(\mu,\beta)/L = \frac{\pi c}{6\beta} - \frac{\mathsf{k}\mu^2}{2\pi\beta}.$$

   Under the continuation stated in the manuscript, $\mu \mapsto -i\beta\mu_{\text{phys}}$, the grand potential is

$$\psi(\mu_{\text{phys}},\beta) = \frac{\pi c}{6\beta} + \frac{\mathsf{k}\beta\mu_{\text{phys}}^2}{2\pi},$$

not the expression proportional to $\mathsf{k}\mu_{\mathrm{phys}}^2/(2\pi\beta)$ in equations (S9) and (S11). The saddle in equation (S10) gives the same result and retains the factor of β multiplying the charge-density term. As printed, these equations are both algebraically and dimensionally inconsistent. The fixed-charge to chemical-potential map and every use of ensemble equivalence should be rechecked after this correction.

The gauge convention also needs a transparent derivation. Equations (S3)–(S5) give holonomy 2μ and $A = \bar{A} = 2\mu\, dt_E/\beta$, while the projector is $e^{i\mu Q}$ and equations (S15)–(S19) use a phase $e^{i\mu q}$ and spectral flow with period 2π. With $q = q_L + \bar{q}_R$ and the field redefinition (S13), transport around the thermal circle appears to supply a phase $e^{2i\mu q}$. The manuscript may intend a compensating convention for the boundary field or for the two chiral charges, but it is not stated. The Wilson-line normalization, large gauge transformations, KMS boundary condition, and shifted Matsubara frequencies should be derived in one common set of conventions before the Fourier projection is repeated.

4. **The radial solution and holographic normalization require a fresh derivation.**

   The near-boundary indicial equation following (S20) is immediate. For $R \sim (1-z)^\alpha$ it gives

$$\alpha(\alpha-1) = \frac{M^2}{4}, \qquad \alpha = \frac{\Delta_\pm}{2}, \qquad \Delta_\pm = 1 \pm \sqrt{1+M^2}.$$

   Equations (S21) and (S23) instead use powers $(1-z)^{2\Delta_\pm}$. Correspondingly, the hypergeometric parameters should obey

$$a + b = 2A + \Delta_-, \qquad ab = (A + \Delta_-/2)^2 + K^2/4,$$

   rather than the formulas in (S22). For the massless example the normalizable mode behaves as $(1-z)^1$, whereas equation (S23) displays $(1-z)^4$. Although setting $\Delta_- = 0$ hides part of the error in some intermediate parameters, it does not repair the transformed solution. Equations (S21)–(S32) cannot serve as a derivation of the quoted Green function until this is corrected.

   There is also a missing normalization in the boundary flux. For the BTZ metric (S2),

$$\sqrt{h}\, n^z = 2r_0^2 z.$$

   Equation (S24) replaces the boundary term by $-zR^*\partial_z R$ without displaying a compensating field or source normalization. This removes a factor proportional to $r_0^2 \sim \beta^{-2}$. Consistently, equation (S26) is dimensionless as a momentum-space correlator and equation (S32) scales as β^{-2}. A dimension-two CFT operator has a thermal coordinate-space two-point function scaling as β^{-4}, so its factorized four-operator diagnostic scales as β^{-8}, not β^{-4} as in equation (18). Contact terms cannot correct this separated correlator scaling, and an operator normalization cannot depend on the temperature. The holographic renormalization, including counterterms and the source normalization, should therefore be redone explicitly.

5. **The projected correlator bounds are internally inconsistent.**

   Even if equation (S32) is provisionally accepted, inserting thermal period 2β gives

$$\left|G_E^{(2\beta)}(\beta,0)\right|^2 = \frac{4\pi^2}{\beta^4}\, f(\nu).$$

   The prefactor used in equation (S34) is instead $\frac{2}{\pi}(4\pi/\beta)^4 f(\nu)$, larger by a factor 128π. Equation (S37) has the analogous mismatch. No normalization change is introduced between these equations.

There are independent errors in the charge projection. Combining equation (S7) with $f(s) = \sum_{\ell\in\mathbb{Z}} f_\ell e^{2\pi i\ell s}$, the normalized Fourier sum at inverse temperature β' has the structure

$$f_0 + 2\sum_{\ell\geq 1} f_\ell \cosh\left(\frac{\pi\beta' qQ\ell}{\mathsf{k}L}\right)\exp\left(-\frac{\pi\beta' q^2\ell^2}{2\mathsf{k}L}\right).$$

For Rényi-2 one must use $\beta' = 2\beta$. Equation (S34) instead uses the $\beta' = \beta$ arguments and omits the factor of two from pairing ℓ and $-\ell$. Consequently the equality in (S34), the corresponding Wightman normalization, and the numerical statement in equation (18) are incorrect as written. Positivity of the Fourier coefficients suggests that a qualitative nonvanishing result may be recoverable, but the authors must redo the complete calculation and control its thermodynamic limit.

Finally, equation (18) asserts $C^W \geq R^{(2)}$. The supplement derives separate proposed lower bounds; those bounds do not establish an ordering between the correlators. This inequality requires an independent operator inequality or should be removed.

6. **The calculation does not establish the headline wormhole claim.**

   The explicit analysis uses an ordinary connected Euclidean BTZ solid torus with a flat U(1) holonomy and applies large-N factorization. It does not construct a new two-copy wormhole saddle created by the charge projection, compare connected and disconnected saddles, or calculate the action of the 'opaque brane' drawn in Figures 2–3. No brane action, tension, boundary conditions, junction conditions, or saddle dominance is specified. Moreover, equation (19) says that a charged operator crosses a projector while changing the sector $Q \to Q + q$; it does not by itself imply an interface through which charged matter cannot propagate. The Wilson defect actually used in the calculation produces a holonomy and an Aharonov–Bohm phase, not the proposed opaque wall.

   The thermofield-double language also needs to distinguish two constructions. The normalized vectorization $|\rho_\beta\rangle\rangle$ relevant to Rényi-2 is TFD-like at inverse temperature 2β, whereas $|\sqrt{\rho_\beta}\rangle\rangle$ relevant to the Wightman correlator is the standard TFD at β. For a fixed-charge state, the representation and opposite orientation of the conjugate copy must also be specified. The paper should then identify which operator pairing is genuinely cross-boundary and which bulk solution prepares it.

   A nonzero neutral product of cross-copy correlators is compatible with an ordinary thermal purification and is not, by itself, a unique diagnosis of a wormhole saddle; the same clustering mechanism applies in non-holographic systems. The abstract and Introduction say that the paper 'shows' that SWSSB is realized through wormholes, while the body more cautiously offers a 'possible resolution.' The authors should either construct and compare the relevant doubled saddles, including the realization of the asymptotic gauge symmetries, or consistently present the wormhole connection as a conjectural interpretation of the factorized thermal channel.

7. **Ensemble equivalence does not presently support the geometric replacement in Figure 3.**

   Equations (S6)–(S11) address thermodynamic functions only. The manuscript explicitly states that equivalence of correlation functions requires further information and defers it to a citation whose title is merely 'In Progress.' Yet the discussion around Figures 2–3 uses this unproved step to replace a fixed-charge projected state by a grand-canonical 'charged black hole.' Local ensemble equivalence requires a thermodynamic sequence $Q(L)$, strict concavity/differentiability, a finite saddle chemical potential, clustering, and control of the imaginary-time-evolved quasi-local operator. It is also a statement about linear local observables, whereas the Rényi order parameter is nonlinear and is designed to retain information about strong versus weak symmetry.

The authors should prove the particular correlator equivalence required in this model, or remove it as evidence for a change of bulk topology. In the Einstein–Chern–Simons model the metric remains BTZ and the charge is encoded in a flat connection; calling the result a charged black hole without this qualification invites an incorrect Einstein–Maxwell interpretation.

8. **The ensemble-emergence argument is formal and, as written, not an ensemble average with a probability measure.**

   Equation (20) mixes an integral over Haar measure with a factor $1/|G|$; for a continuous group such as U(1), $|G|$ is not defined in this manner. More substantively, equations (20)–(22) decompose the identity $\operatorname{Tr}\rho = 1$ for a specially twirled state. A convex decomposition of one density matrix into symmetry-related pure states is not automatically a disorder average over theories or the gravitational ensemble associated with Euclidean wormholes. The claimed equivalence between these notions must be argued rather than assumed.

   Equations (24)–(26) have additional technical problems. The vector in (24) is not generically the vectorization of a Hermitian positive density matrix without conjugation and positivity constraints on the couplings. The proposed Hubbard–Stratonovich weight has a positive indefinite bilinear exponent, unspecified integration contours, and charge labels that do not manifestly make the couplings neutral. It is therefore generally complex or nonnormalizable, not a probability distribution. The authors should specify the contours and reality conditions, verify the decoupling algebra and symmetry transformations, and distinguish a formal Schwinger–Keldysh auxiliary-field measure from a positive statistical ensemble. Otherwise this section should be considerably shortened and labeled as a speculative analogy.

9. **The swampland discussion conflates state symmetry, boundary global symmetry, bulk gauge symmetry, and replica bookkeeping.**

   The usual no-global-symmetry statement concerns the operator content of a theory of quantum gravity. Strong and weak symmetry in this paper instead describe how a particular density matrix transforms. The worked AdS/CFT example has a boundary global U(1) dual to a bulk gauge field, which is compatible with the usual quantum-gravity statement and does not directly test a prohibition on bulk global symmetries. The proposed Swampland SWSSB Conjecture therefore needs a precise bulk/boundary algebraic formulation, a thermodynamic limit, and a definition of 'strong SSB' before it is falsifiable.

   The species-bound argument in the final paragraph is not valid as presented. Schwinger–Keldysh or replica contour copies are bookkeeping copies of one theory; increasing the number of contour segments does not create n independent physical massless vector species in one effective field theory. In real-time holography, the bulk fields on different contour segments are glued analytic continuations of the same fields. A species bound therefore cannot constrain the replica number without a separate construction of independent normalizable modes. This argument should be removed or rebuilt from an explicit multi-boundary bulk theory.

10. **The novelty claim should be focused on what is actually new and controlled.**

    SWSSB diagnostics, the fixed-charge versus grand-canonical distinction, Wilson-line charge projection, thermal BTZ Green functions, ensemble-emergent symmetries, and general links between wormholes and ensembles all have substantial antecedents cited in the manuscript. The potentially publishable new contribution is narrower but still interesting: an explicit fixed-charge holographic Rényi/Wightman SWSSB calculation and its careful geometric interpretation. The revised paper should compare this contribution directly with the cited mixed-state SWSSB work, symmetry-resolved AdS_3/CFT_2, standard BTZ correlator calculations, and the literature on average symmetries.

The lattice discussion should be identified as motivation rather than a result of this paper. Likewise, calling SWSSB a 'critical phenomenon' is unsupported here because no critical point, transition, or critical scaling is analyzed.

Minor comments

1. Euclidean BTZ is a solid torus: the Euclidean-time cycle contracts at the horizon. It is not globally a product $[0, \beta] \times T^2$, as stated in the Holographic SWSSB section. This should be corrected because the contractible and noncontractible cycles are central to the holonomy argument.

2. Equation (1) requires an explicit convention for vectorization and for the transpose or complex conjugation of minus-copy operators. With $|\rho\rangle\rangle = \sum_i p_i |\psi_i\rangle|\psi_i^*\rangle$, the trace formula in equation (3) follows only after those conventions and operator orderings are fixed.

3. Spatial smearing of a continuum field does not generally make it a bounded operator; a smeared scalar field still has infinite operator norm. The footnote following equation (3) should use bounded observables, such as appropriate Weyl operators, or state an energy-cutoff/domain prescription.

4. The assertion about thermal AdS is not supported by a calculation, and thermal AdS is outside the decompactified regime in which the manuscript says BTZ always dominates. Either provide the corresponding projected correlator analysis or omit the claim.

5. The discussion of finite N and finite volume should not treat a nonzero correlator as evidence for strict long-range order. At finite volume there is no infinite-separation limit, and an exact symmetry is restored after summing all saddles unless a symmetry-breaking order of limits is specified.

6. Equation (S22) states $A > 0$, but $A = |W_n|/2$ vanishes for zero Matsubara modes at spectral-flow points. Those modes and the continuity at $\nu = 0, 1$ require separate treatment.

7. The symbol q is used both for the probe charge and, in equation (S10), for the charge density Q/L. Distinct notation would prevent confusion in the subsequent Fourier formulas.

8. A central gap is currently assigned to a bibliography entry titled 'In Progress.' This cannot substitute for a proof in the submitted paper. The incomplete lecture-note citation and duplicate bibliography entries should also be cleaned up.

Recommendation

I recommend **major revision**. The qualitative observation that a fixed-charge thermal state can exhibit a nonzero factorized Rényi/Wightman channel is potentially interesting and may be publishable. Before that conclusion can be assessed, however, the authors must correct the lattice statement, redo the BTZ scalar and charge-projection calculation with consistent normalization, establish the intended thermodynamic regime, and either demonstrate the relevant wormhole saddle or substantially narrow the geometric claim. The ensemble and swampland sections should likewise be made mathematically precise or clearly presented as conjectural outlook. Because these changes affect the central quantitative result and the headline interpretation, they go well beyond a minor revision.