Referee Report

on *Low-Temperature Holographic Conductivity*

arXiv:2607.12064

Summary and overall assessment

The paper studies the low-temperature electrical response of a finite-density Reissner–Nordström AdS_4 black brane. It first reviews a matched-asymptotic calculation of the transverse Maxwell perturbation and identifies its near-AdS_2 mode with an operator of dimension $\Delta = 2$. The authors then replace the semiclassical IR correlator by the corresponding Schwarzian-dressed two-point function. Their main claim is that the resulting dissipative conductivity is non-monotonic: it falls as the temperature is lowered, reaches a minimum near $CT \simeq 0.023$, and subsequently grows as $(CT)^{-1/2}$. A second calculation attempts to obtain the leading correction directly from the four-dimensional one-loop determinant of lifted tensor, vector, and gauge zero modes. This gives terms proportional to T and $T \log T$, which are said to resemble the high-CT expansion of the Schwarzian result qualitatively but not to agree with it quantitatively.

The subject is timely and potentially interesting. Applying exact Schwarzian matter correlators to charge transport, and trying to connect that calculation to higher-dimensional nearly-zero modes, could make a useful contribution. The manuscript also usefully emphasizes the non-commuting low-frequency and low-temperature limits. However, the central observable is not presently identified correctly, several displayed equations needed for the headline result are mutually inconsistent, and the four-dimensional calculation lacks essential steps connecting a Euclidean determinant to a retarded correlator. Some of these problems are exact algebraic errors; others concern the physical definition and control of the approximation. In its present form the paper does not establish either the claimed DC behavior or the proposed comparison between the two approaches at the standard required for publication.

Recommendation

I recommend **major revision**. The Schwarzian calculation may contain a publishable result for the regular (or incoherent) part of the conductivity at finite spatial volume, provided the authors correct the equations, fix the normalization and boundary conditions, and sharply delimit the observable and order of limits. The four-dimensional section requires a substantially more complete derivation; if the issues below cannot be resolved, it should be recast as a clearly labeled exploratory calculation rather than a second determination of the conductivity.

Major comments

1. **The total DC conductivity is infinite in the system studied.**

   The background is translationally invariant and has nonzero charge density. Consequently the electric current overlaps with conserved momentum, and the optical conductivity contains a zero-frequency delta function (equivalently, an i/ω pole). Equations (2.38)–(2.39) and (3.9)–(3.10) can at most describe the regular dissipative part, often denoted $\sigma_{\rm reg}$ or σ_Q, after the momentum contribution has been separated. They are not the full DC conductivity. This distinction is especially important for the interpretation in Section 5: the statement that the resistivity vanishes and the suggestion of a new low-temperature phase do not follow from a divergence of the regular part while the full conductivity already has an infinite Drude contribution at every temperature.

The paper should derive the relevant Ward-identity decomposition, display the Drude term, and state precisely which current and which conductivity are being computed. One clean option is to formulate the calculation for the incoherent current orthogonal to momentum. Another is to add momentum relaxation, but that would be a different model. At minimum, all occurrences of "DC conductivity" and "resistivity" must be revised consistently, and the physical claim in equations (5.1) and the surrounding text must be reconsidered. The companion momentum-relaxing calculation mentioned in Section 5 does not remove this issue from the present paper.

2. **The formula supporting the reported minimum is internally inconsistent.**

   Let $x = CT$. As printed, equation (3.10) contains the relative expansion

   $$x^{-1/2}\left[1+\frac{2}{3}(\pi^2+24)x-\frac{2}{15}(\pi^2-240)x^2+\cdots\right].$$

   Differentiating this expression gives

   $$6(\pi^2-240)x^2-10(\pi^2+24)x+15=0,$$

   not equation (3.11). It would predict $x_{\rm min} \simeq 0.0383$ and a different minimum value. Direct expansion of equation (A.6), using the exact $\Delta = 2$ gamma-function product at $\omega \to 0$, instead gives the quadratic coefficient

   $$-\frac{2\pi^2}{15}(\pi^2-240)x^2.$$

   With this additional factor of π^2, equation (3.11), $CT_{\rm min} \simeq 0.0230$, and the number 0.117194 in equation (3.12) become mutually consistent. Thus the headline location may survive, but the principal conductivity formula is wrong as written.

   The corrected expression should be derived explicitly from the full correlator and checked by minimizing the unexpanded equation (A.6), and preferably the original numerical integral (3.3), rather than only a truncated low-x series. The error estimate in Appendix A is shown only for $C\omega = 10^{-8}$ and $0.005 \le CT \le 0.1$; it does not by itself establish the asymptotic $T \to 0$ claim or the frequency dependence in Figure 4. The temperatures used in Figure 4 should be stated, and the plotted range should be checked against the stated conditions $C\omega \ll CT, (CT)^2$.

   There is a separate normalization error in equation (3.13): equation (2.37) already relates $G^{\rm R}_{\rm UV}$ to the IR correlator with the factor $2L^2/(\mu^2\kappa_4^2)$. Equation (3.13) multiplies $\operatorname{Im} G^{\rm R}_{\rm UV}/\omega$ by that factor again. Either the correlator in (3.13) should be $\mathcal{G}^{\rm R}_{\rm IR}$, or the extra factor should be removed. As written, the optical conductivity and Figure 4 double count the UV/IR conversion.

3. **Several equations in the matched-asymptotic analysis are exactly incompatible and require a new matching calculation.**

   First, the frequency term in equation (2.15) is written as $\Omega^2/(1-Y^2)$. The ingoing solution (2.16) and the repeated equation (3.1) require instead $+\Omega^2/(Y^2-1)$. With the printed sign in (2.15), the near-horizon indices are not the oscillatory indices used in (2.16).

Second, equation (2.29) does not follow from equations (2.4a) and (2.5). With $u = r/r_0$, those equations give

$$Y = \frac{3r_0}{\pi L^2 T}(u-1) - 1 - \frac{2\pi L^2}{3r_0}T + O(T^2),$$

whereas equation (2.29) omits the constant -1 and has a different coefficient for the term linear in T. Substituting the outer horizon into the printed relation already fails to give $Y = 1$. Because (2.29) is used to obtain equations (2.32)–(2.35), the matching must be redone and the effect on the UV/IR relation quantified. The appearance of the arbitrary overlap coordinate $\log(u-1)$ in the supposedly constant coefficient a_O in equations (2.33) and (2.35) also needs an explicit matching-scale cancellation.

Third, equation (2.23) has the constraint coefficient $4\pi L^4 T/(\sqrt{3}r_0^2)$, consistent with equation (2.13) and the coordinate change, while equation (3.2) replaces it by $4\pi L^2 T/(\sqrt{3}r_0)$. These expressions have different dimensions. These are not harmless presentational variations: the matching and the later four-dimensional eigenvalue shifts use the associated metric perturbation.

4. **The boundary condition for the metric perturbation is not the one appropriate to a pure electric-current correlator.**

   For a $J_y J_y$ correlator the boundary electric source should be varied while the non-normalizable boundary metric source is held at zero. Equation (2.28) displays that metric source explicitly. Instead of setting it to zero, the manuscript sets the integration constant H_{in} to zero at the horizon and obtains equation (2.41). Regularity does not, without further argument, require the constant value of ${h^y}_t$ at the horizon to vanish; moreover, equation (2.41) generically leaves a nonzero boundary h_{ty} source. The commented alternative immediately before equation (2.40) would impose a different condition.

   This issue does not change the decoupled Maxwell ODE at the classical level, but it determines which mixed current–momentum source is being varied. It is particularly consequential in Section 4, where h_{ty} itself is inserted into the lifted eigenvalues. The authors should impose ingoing/regular horizon conditions and zero boundary metric source simultaneously, discuss any residual diffeomorphism, and recompute the four-dimensional source dependence. Until this is done, the determinant calculation is not demonstrably a $J_y J_y$ correlator.

5. **The normalization of the Schwarzian operator and the coupling C must be fixed, not suppressed.**

   Equation (2.37) is introduced with a footnote saying that a numerical prefactor is ignored, yet equations (3.10)–(3.12) make absolute claims about the conductivity, its chemical-potential dependence, and its minimum. The exact Schwarzian correlator (3.3) uses a definite convention for the operator $\mathcal{O}_{\Delta=2}$, while the ratio of normalizable to non-normalizable Maxwell modes has a normalization fixed by the reduced bulk action. A multiplicative matching coefficient is therefore required. It should be obtained by an explicit two-dimensional reduction or by matching the $CT \gg 1$ Schwarzian correlator to the semiclassical result (2.39). Merely replacing the scaling form in (2.36) by (3.8) does not determine this coefficient.

   There is already a factor-of-two inconsistency inside the stated Schwarzian formulae. The integral in equation (3.3) contains the factor $2k$, but equation (3.5) defines $g_T(k)$ with only k while leaving the overall prefactor unchanged. The mismatch is repeated when the formula is rewritten in equation (A.1). The authors should state which convention was used to obtain (A.6) and propagate a single normalization through the UV correlator.

There is also an apparent inconsistency in the value of C. From the action (2.1), $\kappa_4^2 = 8\pi G_4$, the torus area $4\pi V_2$, and the horizon area, one obtains

$$S_{\rm BH} = \frac{8\pi^2 V_2 r_+^2}{\kappa_4^2 L^2}, \qquad C = \frac{8\pi V_2 r_0}{3\kappa_4^2}$$

by matching the linear entropy to $S = S_0 + 4\pi^2 CT + \cdots$. The manuscript instead writes $C = 4V_2 r_0/3$ while retaining κ_4 elsewhere. If special gravitational units are being used in the entropy discussion, they must be stated and then applied consistently. Restoring these factors is essential for the loop-counting interpretation and for comparing Sections 3 and 4.

Finally, $C \propto V_2 r_0$ and $r_0 \propto \mu$ at fixed chemical potential convention. Thus varying μ, C, and V_2 independently in Figure 3 is not generally compatible with the background family. The ensemble and the quantities held fixed should be specified.

6. **The reduction to one neutral $\Delta = 2$ primary needs to be justified at the level required by the claimed universality.**

   The transverse Maxwell perturbation is coupled to the shear metric perturbation at finite density. The constraint permits a decoupled radial equation, but it does not by itself prove that the complete low-energy generating functional is obtained by coupling a canonically normalized neutral primary to the Schwarzian and dressing only its two-point function. In a charged near-AdS_2 reduction there can also be boundary gauge/phase modes, constraints, and mixing with the reparameterization mode. The choice of metric source discussed above makes this more than a formal concern.

   The paper should derive the quadratic two-dimensional action for this gauge-invariant transverse mode, including its boundary term and its coupling to the Schwarzian mode, or cite and apply a derivation that fixes all coefficients and sources in precisely this RN-AdS_4 channel. This would also show whether the standard correlator (3.3) may be inserted unchanged. Without such a derivation, the $(CT)^{-1/2}$ scaling is an interesting ansatz but the assertion in Section 5 that it is universal across broad near-AdS_2 models is not established.

7. **The order of the low-temperature and thermodynamic limits must be part of the statement of the result.**

   The Schwarzian coupling is extensive: $C \propto V_2$. The strongly quantum regime $CT \ll 1$ therefore occurs at $T \lesssim 1/V_2$ and disappears if the planar thermodynamic limit is taken first. Correspondingly, the correction in (3.10) carries inverse powers of C, and the four-dimensional answer (4.27) explicitly carries $1/V$. These are global finite-volume throat fluctuations, whereas conductivity is normally presented as an intensive thermodynamic response.

   The authors should state that the proposed low-T upturn is obtained by taking $T \to 0$ at fixed finite torus volume, explain how the current correlator is normalized by the volume, and discuss the alternative $V_2 \to \infty$ order of limits. This point also constrains the physical interpretation of the minimum and any comparison with macroscopic strongly coupled matter. The non-commutativity of ω and T is discussed carefully; the equally important non-commutativity of T and V_2 should be treated with the same care. Since Section 2 works at fixed chemical potential, the authors should also explain whether the charged near-AdS_2 effective theory and its $U(1)$ phase mode are being treated in the grand-canonical or fixed-charge ensemble.

8. **The four-dimensional Euclidean calculation is not connected consistently to a retarded correlator.**

Section 4 starts from a Euclidean thermal path integral, but equations (4.10)–(4.11) insert a source proportional to $e^{-\Omega\tau}$ for continuous real Ω. Such a source is not periodic on the Euclidean thermal circle except at discrete Matsubara frequencies. The derivation should first be performed for periodic modes and only then analytically continued. No such continuation, or the associated branch choice, is given. A Euclidean determinant with real sources is real, while equations (4.12), (4.17)–(4.19), and (4.26) are described as its "imaginary part."

There is an even more direct symptom. The common source term in equation (4.22) is

$$\int_{-\infty}^{\infty} d\omega\; \omega\, C_{\rm in}(\omega) C_{\rm in}(-\omega).$$

For a real bosonic source the product is even in ω, so the displayed integral is odd and vanishes. Equivalently, only the even part of a quadratic Euclidean kernel contributes. Dissipation can arise after continuing a nonanalytic Euclidean kernel such as $|\omega_n|$, but it cannot be obtained from the expression as written. The authors must supply the Matsubara kernel, carry out the continuation to G_R, and then apply the Kubo formula. Equations (4.5), (4.24), and (4.25) should also be written in terms of derivatives of $\log Z$ for a connected correlator, with the conditions under which first derivatives vanish stated explicitly.

9. **The lifted-mode determinant and its renormalization are not yet a physical one-loop prediction.**

The assertion around equation (4.7) that the first source-dependent eigenvalue shift occurs at $O(\epsilon^2)$ needs proof. In second-order spectral perturbation theory there are generally both a direct second-order operator matrix element and sums over intermediate nonzero modes generated by the first-order perturbation. Equations (4.9) and (4.12) do not show these mixing terms. Similarly, the statement that the off-diagonal metric–gauge operator $O_{\rm int}$ "continues to have zero eigenvalues" is not enough to factor the coupled determinant as in (4.4); the full block operator, gauge fixing, ghosts, mode degeneracies, and the powers $\det^{-1/2}$ versus $\det^{-1}$ must be accounted for.

The radial integrals for the vector and gauge modes diverge at the AdS_2 boundary. This is precisely where the throat approximation must be matched to the outer geometry. Dropping $\log\tilde{\epsilon}$ by minimal subtraction in equations (4.26)–(4.27), without exhibiting bulk and boundary counterterms or an outer-region cancellation, leaves an arbitrary finite answer. Indeed the result retains the normalization length l through $\log(\pi l^2 T/r_0)$. A physical conductivity cannot depend on an arbitrary normalization scale for the zero modes. The wholesale omission of nonzero modes also needs a power-counting argument: it may be appropriate when extracting a universal nonanalytic low-T term, but it is not justified for a complete one-loop conductivity, whose regular finite contribution is scheme- and matching-dependent.

Finally, (4.27) is only a correction, not a complete conductivity. Its bracket becomes negative as $T \to 0$ for fixed l/r_0, and the $T\log T$ term would eventually dominate the positive classical regular contribution proportional to T^2. The paper must identify the controlled temperature window, combine tree and loop terms, demonstrate spectral positivity there, and show scheme independence. Until these points are resolved, the claimed qualitative agreement in Section 4.4 is too weak to support a physical comparison. The classical T^2 term in the large-CT expansion of (3.8) must first be subtracted before comparing quantum corrections, and matching a remaining polynomial T term to $T + T\log T$ is not by itself parametric agreement.

10. **The relation to prior charge-correlator and incoherent-transport results should be made quantitative.**

    The paper cites the recent analysis of near-extremal holographic charge correlators but does not explain which part of the matched solution, low-frequency limit, or UV/IR map is new relative to that work. The statement in Section 1 that the recent finite-low-temperature charge-correlator analysis simply took $T \to 0$ first should be made more precise. It also does not compare equation (2.39) with the standard hydrodynamic expression for the regular conductivity of a finite-density translationally invariant state. Such a comparison would help identify the correct observable and provide a nontrivial check of all normalizations. The novelty of Section 3 appears to be the exact Schwarzian dressing of this particular current channel; that should be isolated clearly from earlier Schwarzian/JT calculations of resistivity, optical response, and shear correlators, where the same non-monotonic $1/\sqrt{T}$ mechanism has already appeared. The claimed higher-dimensional extension should likewise be distinguished from the authors' cited one-loop shear calculation by spelling out which new gauge and vector-mode ingredients have actually been controlled.

Minor comments and presentation

1. Equation (2.37) says a numerical prefactor is irrelevant "for the final result," but the subsequent results use an absolute normalization. The complete coefficient and dimensions should be shown at every step.

2. The phrase below equation (2.37), "with the appropriate IR Green's function This relationship," is missing punctuation. The manuscript should also use one notation consistently for $G_{\mathrm{R,IR}}$ versus $\mathcal{G}_{\mathrm{R,IR}}$.

3. The Wightman convention in equation (3.3), including the prefactor $1/\pi$, and the sign and factor $1/2$ in equation (3.7) should be checked against the authors' retarded-function convention. The continuation for negative frequency and the KMS relation should be stated.

4. The discussion below equation (3.8) quotes an absolute error of order 10^{-15}, while Appendix A quotes a relative error of order 10^{-7}. These are not contradictory, but the main text should identify which error measure is being used rather than presenting the former as an accuracy statement by itself.

5. Figure 4 has no legend or numerical temperatures, and its caption only says that temperature decreases from top to bottom. All parameter values and the validity range of the approximation should be supplied. Figures 2 and 3 should likewise specify which overall normalization has been divided out.

6. In the Euclidean discussion, equation (4.9) uses $\sqrt{-g^{(0)}}$ rather than the Euclidean measure. The notation also alternates among L and l, V and V_2, and κ and κ_4 without always defining the relation. This is particularly dangerous because l survives in (4.27).

7. Equations (4.8), (4.20), and (4.21) use inconsistent conventions for whether the explicit factor of T is included in $\delta\Lambda(g^{(1)}, A^{(1)})$. A uniform definition is needed before applying the product (4.23).

8. Equations (4.3), (4.7), and the Section 2 background expansion alternately use $g^{(1)}$, $Tg^{(1)}$, and $\lambda g^{(1)}$. The perturbative order of every source and near-extremal correction should be made uniform.

9. The assertion in Appendix A that the large-k replacement is adopted "without further error quantification" is followed by numerical error quantification. The text should instead explain precisely what was tested and what remains an extrapolation.

10. The bibliography contains a duplicate entry for the one-loop shear-viscosity paper. The companion-paper entry still contains the placeholder arXiv identifier "2607.xxxxx" and should be updated. More generally, unpublished and companion results should not be needed to define the observable in this paper.

11. Several broad statements should be narrowed after the technical issues are resolved. In particular, the existence of a minimum in a regular transport coefficient is not evidence by itself for an instability or phase transition, and the phrase "quantum membrane paradigm" is premature without a demonstrated relation for the complete response function.

Conclusion

There is a potentially valuable core idea in the Schwarzian treatment of the transverse current channel, and the corrected expansion appears capable of reproducing the quoted minimum. Nevertheless, publication requires a clean definition of the regular/incoherent conductivity, a corrected and normalized matched calculation with the proper metric source, and an explicit account of finite volume and order of limits. The four-dimensional result needs a valid Euclidean-to-retarded continuation and a regulator-independent determinant calculation. Addressing these points would both strengthen the novelty claim and turn the comparison of the two approaches into a meaningful physical test.