

Referee Report

on *Low-temperature Quantum-corrected Holographic Transport with Momentum Relaxation*

Summary and overall assessment

The paper studies very-low-temperature transport in homogeneous holographic models with explicit momentum relaxation. In the neutral linear-axion background, the authors first obtain the classical UV/IR matching relations for the shear metric perturbation and for a probe Maxwell field. The corresponding near-AdS₂ operators have dimensions $\Delta = 2$ and $\Delta = 1$, respectively. The classical IR correlators are then replaced by exact Schwarzian-dressed correlators, leading to a claimed low-temperature enhancement of the shear spectral coefficient and electrical conductivity. The analysis is subsequently generalized to a charged axion black brane. There the shear dimension varies continuously between one and two, while two master fields of dimensions one and two enter the electric channel. Finally, Section 6 proposes a general argument for a $(CT)^{-1/2}$ low-temperature rise and an intermediate minimum whenever $\Delta > 1$.

Extending near-AdS₂ transport calculations to explicit momentum relaxation is timely and physically well motivated. The varying shear dimension could test the Schwarzian mechanism, while a controlled small-energy analysis may support a universal $(CT)^{-1/2}$ enhancement in an appropriate finite-volume window. However, the present manuscript does not yet establish its principal quantitative claims. The canonical Schwarzian operator is not normalized to the bulk coefficient-ratio Green function; the displayed shear formula is algebraically inconsistent with the preceding correlator; one of the two electric master fields is incorrectly discarded after quantum dressing; and the arbitrary- Δ proof applies a large- k expansion precisely in a small- k regime. These issues affect the central plots and, in the charged case, even the asserted monotonicity of the conductivity.

The classical matched-asymptotic structure is broadly plausible and the core physical idea is potentially publishable. The quantum calculation nevertheless requires a substantial revision, including corrected analytic formulae, renewed finite-density numerics, and a sharper definition of the observables and limits.

Recommendation

I recommend **major revision**. I would be willing to reconsider a version that fixes the IR normalization from a controlled reduction or a semiclassical matching condition, restores both master-field contributions in the charged conductivity, replaces the Section 6 argument by a valid small- k analysis, and regenerates all affected figures. The paper should also distinguish the full DC response from its regular part and delimit the finite-volume temperature window in which the Schwarzian result applies.

Major comments

1. **The Schwarzian correlator is not normalized to the bulk IR Green function.** The Green functions that appear in the matched bulk solutions have definite normalizations. In the neutral

electric channel, equation (3.33) gives

$$\mathcal{G}_{\text{IR}}(\omega) = \frac{i\omega}{3},$$

and equation (3.35) gives $G_{\text{UV}} = -3\mathcal{G}_{\text{IR}}$, yielding $\sigma_{\text{DC}} = 1$ in equation (3.37). By contrast, the canonical $\Delta = 1$ Schwarzian correlator used in equations (4.26)–(4.29) has

$$\frac{\text{Im } G_{\Delta=1}^R}{\omega} \longrightarrow 1 \quad (CT \gg 1),$$

so equation (4.29) tends to 3, as stated explicitly in equation (4.30), rather than to the classical value 1. This is a direct semiclassical matching failure, independent of sign conventions. The missing factor is $1/3$ in this channel.

The same problem occurs for shear. The zero-temperature bulk normalization is fixed by equation (3.10), $\mathcal{G}_{\text{IR}} = i\omega^3/81$. Inserting the canonically normalized Schwarzian bilocal without an operator matching coefficient cannot reproduce this quantity. Using the formulae as written but correcting the algebra discussed below, the $CT \gg 1$ shear result exceeds the classical equation (3.20) by a factor $27/2$; a factor $2/27$ is therefore required for the canonical $\Delta = 2$ correlator in this convention. At finite density, equation (5.39) contains the additional channel-dependent amplitudes λ and $\lambda^3/3$, yet these amplitudes disappear when the correlators are replaced in Sections 5.1 and 5.2.

The authors should derive the quadratic near-AdS₂ action and the normalization of each effective operator, including the boundary term that identifies source and response. Alternatively, each multiplicative factor can be fixed by requiring the $CT \gg 1$ Schwarzian correlator to reproduce the finite-temperature semiclassical AdS₂ correlator. This must be done separately for $\Delta = 1$, $\Delta = 2$, and generic $\Delta(\alpha/r_0)$. Appendix B normalizes the UV source/response ratio but does not solve this independent IR operator-normalization problem. Equations (4.19), (4.29), (5.27), and (5.52), and Figures 2–8, must all be rechecked afterward.

2. Equation (4.19) does not follow from equations (3.16) and (4.15)–(4.18).

This is an algebraic error separate from the preceding matching issue. Let $x = CT$ and $D = 4 - 3 \log 3$, and denote by $\mathcal{B}(x)$ the complete square bracket in equation (4.15). Linearizing equation (3.16) in the retarded IR correlator and using equation (4.16) gives

$$\eta_{\text{canonical}}(x) = \frac{64}{9D^2} \frac{G_{\Delta=2}^>(\omega=0)}{2T} = \frac{8}{27C^2D^2} \mathcal{B}(x).$$

The printed equation (4.19) is instead equivalent to replacing the coefficient $16\pi^2x^2$ multiplying the second bracket in equation (4.15) by $24\pi^2x^2$. In particular, the formulae preceding (4.19) give

$$\eta_{\text{canonical}} = \frac{8\sqrt{2}}{27\pi^{3/2}C^2D^2} x^{-1/2} \left[1 + \frac{2}{3}(\pi^2 + 24)x + \cdots \right]$$

for $x \ll 1$, not the coefficient $\frac{2}{3}(\pi^2 + 36)$ in equation (4.20). At large x they give

$$\eta_{\text{canonical}} = \frac{128\pi^2}{27C^2D^2} x^2 \left[1 + \frac{3}{2\pi^2x} + \cdots \right],$$

which is smaller than the large- x expression in (4.20) by a factor $2/3$.

This correction also moves the minimum of the unnormalized expression from the quoted $CT_{\text{min}} \simeq 0.01769$ to approximately 0.02244. The latter number follows directly by minimizing the bracket

in equation (4.15). The authors should identify where the extra factor entered, correct equations (4.19)–(4.21), and regenerate Figures 2 and 3. Only after this internal correction should the separate bulk-to-Schwarzian matching factor in comment 1 be applied.

3. The charged conductivity omits a master field that becomes leading after quantum dressing.

Equations (5.36)–(5.39) correctly identify two electric master fields: Φ_+ has $\Delta = 1$ and $\mathcal{G}_{\text{IR},+} \sim i\omega$, whereas Φ_- has $\Delta = 2$ and $\mathcal{G}_{\text{IR},-} \sim i\omega^3$ in the classical extremal geometry. This classical power counting is then used in expanding the UV relation (5.44): equation (5.48) retains only $\mathcal{G}_{\text{IR},+}$, and equations (5.49)–(5.52) consequently assign the complete DC conductivity to the $\Delta = 1$ correlator.

That ordering is not valid after the proposed quantum replacement. At fixed nonzero temperature, equations (4.16) and (4.18) imply

$$G_{\Delta}^R(\omega) = \frac{i\omega}{2T} G_{\Delta}^>(0) + O(\omega^2)$$

for *both* $\Delta = 1$ and $\Delta = 2$. Thus the Schwarzian-dressed Φ_- correlator is also $O(\omega)$ and contributes to the DC Kubo limit. The replacement must be made in the full expression (5.44) before expanding in ω . Generically the result contains

$$K_+(\alpha) G_{\Delta=1}^R + K_-(\alpha) G_{\Delta=2}^R,$$

where K_- contains the independent-solution determinant $a_{1,-}b_{2,-} - a_{2,-}b_{1,-}$. There is no argument in the paper that this coefficient vanishes; for a genuine independent basis it should not do so generically.

This issue directly affects equations (5.49)–(5.52), Figure 8, and the conclusion that the finite-density conductivity is monotonic because it is controlled by $\Delta = 1$. The $\Delta = 2$ term has a different large- CT dependence and can itself produce a minimum. The charged conductivity must therefore be recalculated with both dressed channels, including their separate normalization factors and the frequency dependence of the source-cancellation coefficients $C_{\pm}$.

4. The finite-density thermodynamic expansion omits a linear term in the stated ensemble.

Equation (5.5c) claims that m_0 receives no correction linear in T when the chemical potential is held fixed. For $d = 3$, however, equations (5.2)–(5.4) give

$$m_0(r_+) = r_+^3 + r_+ \left(\frac{\mu^2}{4} - \frac{\alpha^2}{2} \right), \quad \mu^2 = 12r_0^2 - 2\alpha^2$$

at extremality. Using

$$r_+ = r_0 + \frac{2\pi}{3}T + \frac{2\pi^2}{9r_0}T^2 + \dots$$

therefore yields

$$\begin{aligned} m_0 &= r_0(4r_0^2 - \alpha^2) + \frac{2\pi}{3}(6r_0^2 - \alpha^2)T \\ &\quad + \frac{2\pi^2(12r_0^2 - \alpha^2)}{9r_0}T^2 + \dots \end{aligned}$$

The quadratic coefficient printed in (5.5c) is correct, but the linear term vanishes only at the neutral endpoint $\alpha^2 = 6r_0^2$. The statement below equation (5.6) that the mass scales quadratically is consequently false for a generic charged brane at fixed μ .

There may be a physically appropriate quadratic excitation in the grand-canonical combination $E - \mu Q$, or in a fixed-charge ensemble, but that quantity and its boundary conditions must be defined and computed. The distinction matters for identifying the Schwarzian coefficient and the breakdown scale, and a fixed-chemical-potential charged throat can also carry a boundary phase mode. The authors should correct the thermodynamics, state the ensemble used by the effective theory, and revisit the inference leading to equation (5.7) and the replacement by a Schwarzian-only partition function.

5. The arbitrary- Δ low-temperature argument uses an expansion outside its domain.

Equation (6.3), or equivalently equation (A.8), is derived from the large- k expansion (A.3) together with $\sinh(2\pi k) \simeq e^{2\pi k}/2$. For $CT \ll 1$, however, the exact integral (4.2) is localized at $k \sim \sqrt{CT} \ll 1$. The successively more singular inverse powers in (6.3) therefore diagnose the breakdown of the large- k expansion; they cannot be used term by term to determine the $CT \rightarrow 0$ limit. Appendix A itself acknowledges that (A.8) departs from the exact result for generic Δ . The claimed constant leading behaviors for $\Delta = 3/2$ and $\Delta = 5/2$ in equations (A.11) and (A.13) are likewise artifacts of retaining the large- k approximation at small k .

There is a simple controlled route to the desired result. Taking $\omega \rightarrow 0$ first in the exact equation (4.2), using $\sinh(2\pi k) \simeq 2\pi k$ and expanding the gamma functions about $k = 0$, gives

$$G_{\Delta}^{>}(0) \sim \frac{2^{7/2-2\Delta}\Gamma(\Delta)^4}{\pi^{3/2}\Gamma(2\Delta)} C^{1-2\Delta} \sqrt{CT} \quad (CT \ll 1)$$

for fixed $\Delta > 0$. The KMS factor then makes the Kubo slope proportional to $(CT)^{-1/2}$ for every such Δ . This salvages an important qualitative claim, but for a different reason than the discussion following equation (6.4): the scaling follows from the universal small-energy Schwarzian density of states, not from selecting a Δ -dependent term in a large- k series. Section 6 and the small- CT part of Appendix A should be replaced by this controlled derivation and checked numerically against the exact integral for several noninteger dimensions.

6. The proposed theorem about a universal minimum is stronger than the UV/IR relation supports.

For the fractional map in equation (6.1), if $\mathcal{G}_{\text{IR}}$ itself becomes large at fixed frequency, the full expression generically approaches b/d rather than diverging. In the ordered Kubo limit the relevant statement is instead

$$\mathcal{G}_{\text{IR}}(\omega, T) = i\omega F(T) + \dots, \quad \partial_{\omega} \text{Im } G_{\text{UV}}|_{\omega=0} = \frac{bc - ad}{c^2} F(T) + \dots,$$

assuming $c \neq 0$. Thus $G_{\text{UV}}(\omega = 0)$ retains its real constant a/c ; it is the dissipative slope, not the full UV Green function, that grows. The paragraph following equation (6.4) and equation (6.5) conflate these quantities and the two orders of limits.

Moreover, the coefficient $(bc - ad)/c^2$ can vanish, change sign, or acquire additional temperature dependence; several IR modes can contribute, as the charged electric channel already demonstrates. Positivity and continuity of the physical Kubo coefficient must also be established. A defensible conditional statement would be: if a single dressed IR mode couples with a nonzero, temperature-smooth coefficient of fixed sign, the exact Schwarzian factor produces a low- CT rise; if the same

observable also grows at large CT , continuity implies at least one intermediate minimum. The stronger claims that every $\Delta > 1$ observable must have a minimum and that $\Delta = 1$ uniquely guarantees monotonicity are not proved.

7. The electric and shear observables need more precise physical definitions.

At finite charge density and $\alpha = 0$, exact translations imply a zero-frequency delta function in $\text{Re } \sigma$ and an i/ω pole in $\text{Im } \sigma$. The full DC conductivity is therefore infinite, not zero. The result of Edalati et al. invoked after equation (5.47) and in Appendix C concerns the regular dissipative part after the momentum contribution is separated. The two noncommuting limits in equations (C.2)–(C.3) should be reformulated distributionally, distinguishing the Drude weight, the regular conductivity, and, where appropriate, an incoherent current. The $\alpha = 0$ curve in Figure 8 cannot be described as the full DC conductivity.

Conversely, in Section 3 the probe current is at zero density and has no overlap with momentum. Its finite value in equation (3.37) does not require momentum relaxation; in four bulk dimensions the constant Maxwell conductivity is also closely tied to electromagnetic self-duality. The paragraph after (3.37), the Introduction, and the opening claim of Section 7 should not attribute this particular finite result to the axioms.

Finally, when translations are explicitly broken, the zero-momentum $T_{x_1 x_2}$ spectral coefficient is not automatically the hydrodynamic shear viscosity governing a conserved transverse momentum mode. The final paragraph briefly acknowledges that the notion is modified, but this qualification is needed when the observable is first defined. The authors should state whether they mean a shear spectral coefficient, a viscoelastic response, or an entropy-production coefficient, and explain which absorption or constitutive interpretation survives.

8. The Schwarzian coupling, transverse volume, and range of validity are not consistent or explicit enough.

The manuscript introduces C in Section 4 without deriving it from the black-brane thermodynamics; only much later does Section 5 state $C = V_2/\varepsilon_{\text{gap}}$. As written, this relation appears inconsistent with the volume convention. Section 2 states that the transverse torus has physical volume $V_{\text{phys}} = 4\pi V_2$. From

$$Z \propto e^{S_0 + 2\pi^2 CT} (CT)^{3/2}$$

one obtains $\partial S/\partial T = 4\pi^2 C + \dots$. Combining this with equations (2.11) or (5.7), which define $\varepsilon_{\text{gap}} = 4\pi^2 (\partial s/\partial T)^{-1}$ using an entropy *density*, gives

$$C = \frac{V_{\text{phys}}}{\varepsilon_{\text{gap}}} = \frac{4\pi V_2}{\varepsilon_{\text{gap}}},$$

not $V_2/\varepsilon_{\text{gap}}$, unless V_2 is silently redefined. For the same reason, the logarithmic correction to the entropy density inferred from the displayed partition function is $3 \log(CT)/(2V_{\text{phys}})$, whereas equation (4.22) uses $3 \log(CT)/(2V_2)$. These factors affect the η/s plots and the quoted discreteness scale and must be derived with one volume convention.

There is a second entropy inconsistency. Equation (4.12) gives

$$S = S_0 + 4\pi^2 CT + \frac{3}{2} \log(CT) + \frac{3}{2} - \frac{1}{2} \log(2\pi),$$

before division by the transverse volume. Equation (4.22) hides the known $4\pi^2 CT$ term inside $O(T)$, but the truncated expression is then used in the η/s plots at values of CT where this term

is not small. For example, at $CT = 0.17$ it is approximately 6.7. Figures 3 and 6 and any claimed minimum of η/s should be recomputed with the complete entropy consistently derived from the same partition function.

This also exposes an important order of limits. The Schwarzian coupling is extensive, $C \propto V_{\text{phys}}$. The regime $CT \ll 1$ is therefore a finite-volume regime at $T \lesssim 1/V_{\text{phys}}$ and collapses to $T = 0$ if the planar thermodynamic limit is taken first. Since conductivity and viscosity are normally intensive thermodynamic responses, the paper must state that the upturn is obtained by lowering T at fixed finite torus volume and explain the alternative order of limits. Comparisons with macroscopic correlated materials are premature without this qualification.

The actual controlled window should be stated in one place. It includes

$$e^{-2S_0/3} \ll CT \ll 1, \quad \omega \ll T,$$

as well as T below the UV scales set by α , μ , and r_0 . The authors should verify that every reported minimum lies above the discreteness cutoff (4.23), and that the $CT \gg 1$ portions used for matching remain within the near-extremal throat regime.

9. The retarded-function conventions and Kramers–Kronig step are inconsistent.

The Wightman transform in the footnote to equation (4.2) contains a nonstandard factor $1/\pi$, while equations (4.16)–(4.18) do not explain whether the retarded transform uses the same convention. The sign of the KMS relation must also be aligned with the convention for $G_R(t) = -i\theta(t)\langle [O(t), O(0)] \rangle$. Across equations (3.1), (3.36), (4.19), (4.29), (5.46), and (B.6), the Kubo formula alternates among iG/ω , $+\text{Im } G/\omega$, and $-\text{Im } G/\omega$. Equation (4.28) labels its left-hand side as an IR correlator even though the text calls it UV, and it has the opposite sign from the relation (3.35). Equation (4.19) likewise writes $\text{Im } G_{\text{IR}}^R$ after purportedly performing the UV/IR conversion.

There is also no valid inference from equation (4.17) to equation (4.18). Knowing only that $\text{Im } G_R$ is linear at small ω does not determine $\text{Re } G_R$ through a dispersion integral over the full spectrum. A subtraction constant and analytic even powers can remain. This may not alter a leading dissipative Kubo slope, but a real IR contact term can change the constants in a fractional UV/IR map unless its scheme is fixed consistently. The paper should avoid reconstructing the real part from incomplete low-frequency data, state a single Fourier and spectral convention, and verify spectral positivity and the classical limits in that convention.

10. The finite-density calculation and the effective reduction are not reproducible at present.

Sections 5.1 and 5.2 depend on numerical connection coefficients $a_{\{1,2\},\pm}$ and $b_{\{1,2\},\pm}$, but the manuscript gives no integration algorithm, horizon and boundary cutoffs, basis normalization, precision, convergence test, or numerical data. These coefficients control the complete α/r_0 dependence in equations (5.27), (5.49), and (5.52). The revision should provide enough information to reproduce them, demonstrate invariance under rescaling either independent outer solution, and test both endpoints $\alpha = 0$ and $\alpha = \sqrt{6}r_0$. The special prescription $C_- = 0$ at the latter endpoint should be obtained as a controlled limit rather than imposed after a divergence.

More fundamentally, the paper assumes that each mixed metric–gauge–axion master field can be treated as an independently and canonically normalized neutral matter primary coupled only to the Schwarzian mode. This is not established by the radial equations alone. A charged near- AdS_2 reduction can contain boundary gauge/phase modes and constraints, and the source condition

(5.45) mixes the two master fields. A quadratic two-dimensional reduction, or a precise existing derivation applied to these gauge-invariant channels, is needed to justify the replacement rule and to fix all normalizations. This derivation should be performed before interpreting the numerical curves as quantum-corrected four-dimensional transport.

11. The novelty and literature comparison should be sharpened.

The classical background, the finite-density DC formula, and the low-temperature classical shear scaling are inherited from the cited works of Andrade–Withers and Hartnoll–Ramirez–Santos. The minimum and $(CT)^{-1/2}$ enhancement of Schwarzian-dressed shear correlators are also central results of the recent papers cited in the Introduction. The genuinely interesting new question here is whether those effects persist with explicit momentum relaxation, especially for the continuously varying shear dimension and the coupled finite-density electric channel. The manuscript should present that as the novelty rather than suggesting that the minimum itself is newly discovered.

The comparison to the recent literature must also address its differing normalizations and conclusions, rather than treating all cited quantum- η/s analyses as interchangeable support. In particular, boundary-correlator and matter-backreaction treatments are directly relevant to the replacement rule used here. A quantitative comparison would provide nontrivial checks of both the high- CT limit and the low- CT coefficient.

Finally, reference [81], *Low-Temperature Holographic Conductivity*, still contains the placeholder identifier 2607.xxxxx. That paper is arXiv:2607.12064 and has three authors in common with the present manuscript. The reference must be corrected, and the authors should delineate explicitly which Schwarzian conductivity derivations, limits, and conclusions are inherited from that companion paper and which are new here. This is particularly important because Section 7 invokes the companion result as evidence for universality.

Minor comments and presentation

1. Equations (3.35) and (5.27) retain red-colored signs or prefactors, apparently from an internal revision. All editorial markup should be removed from a submitted version.
2. Equation (3.19) inserts a factor $3r_0^2$ that is absent from equation (3.18). It does not change the zero limit, but its origin and the dimensions should be explained before equation (3.20) is used as a normalization check.
3. In the discussion of equation (5.10), the two independent Hankel functions are both written as $H_\nu^{(1)}$; the second should be $H_\nu^{(2)}$.
4. The text preceding equation (5.17) describes the AdS_2 -boundary overlap using $r \rightarrow \infty$. In the original four-dimensional coordinate it is the region $\omega \ll r - r_0 \ll r_0$, not the asymptotic AdS_4 boundary. The limits should be labeled unambiguously.
5. Equation (5.25) uses the symbol Δ in the large- r falloffs of the shear mode, while the same subsection uses $\Delta = 1/2 + \nu$ for the IR AdS_2 dimension. The UV shear source and response behave as a constant and r^{-3} ; a different symbol or the explicit powers would avoid confusion.
6. Figure captions 5–8 should state the numerical frequency, radial cutoffs, values of C and S_0 , and

the precise normalization of the plotted correlator. The figures should also display or quote the expected analytic endpoint limits after the recalculation.

7. Appendix B alternates between coefficients defined in the physical r expansion and coefficients in $u = r/r_0$. Since $r^{-3} = r_0^{-3}u^{-3}$ and $r^{-1} = r_0^{-1}u^{-1}$, equations (B.17), (B.18), (B.32), and (B.33) should specify which coefficient is used and retain the necessary powers of r_0 .
8. The entropy obtained from equation (4.12) also contains a temperature-independent term $3/2 - \frac{1}{2}\log(2\pi)$ in addition to $\frac{3}{2}\log(CT)$ and the term linear in T . The authors should say which normalization absorbs this constant before using the entropy to locate a zero.
9. The paper should consistently distinguish a Wightman function, a spectral density, an IR retarded function, and the UV retarded function. At several points G_Δ , G_{IR} , and G_{UV} are interchanged in prose even when their normalizations and signs differ.
10. The statement below equation (4.20) about the $\Delta = 1$ dependence and hence that of η should instead refer to the conductivity; the shear channel in that discussion has $\Delta = 2$.
11. The abstract and conclusion should describe the low-temperature enhancement as occurring only inside the controlled window, not as a literal $T \rightarrow 0$ divergence. Claims of similarity to correlated materials should be limited to the shape of a finite-volume theoretical curve unless an intensive thermodynamic limit is supplied.
12. The organizational paragraph in Section 1 omits Section 6. The bibliography should replace all placeholder metadata, and the manuscript should be proofread for malformed characters, inconsistent capitalization, and several grammatical errors. These are secondary to the physics issues above but currently impede readability.

Conclusion

The manuscript contains a promising application of near-AdS₂ quantum dynamics to momentum-relaxing holography. In particular, a direct small- k treatment of the exact Schwarzian integral appears to support a universal low- CT Kubo enhancement, and the finite-density shear channel offers a nontrivial continuously varying dimension. The present quantitative results are nevertheless not reliable: the IR operator normalization fails a basic classical check, equation (4.19) contains an independent algebraic error, the finite-density thermodynamics is inconsistent in the stated ensemble, and the charged conductivity omits a quantum-leading $\Delta = 2$ contribution. Correcting these issues may preserve part of the qualitative picture, but it will require new analytic formulae, new numerical curves, and a more conditional universality statement. A substantially revised manuscript addressing the points above could become suitable for publication.