Referee Report

"Composite worldline instantons and the nonperturbative particle decay in constant external electric and magnetic fields"

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Recommendation: Major revision.

Summary of the paper

The manuscript studies field-induced decay of a charged particle into charged and neutral daughters in cases where the corresponding particle decay is forbidden kinematically in the absence of the field. Its main methodological proposal is that such processes can be described by a composite worldline instanton assembled from the Euclidean trajectories of the initial and final particles. The authors use scalar particles and retain only the leading semiclassical exponential, on the grounds that spin and vertex structure affect the prefactor rather than the classical action.

For a constant electric field, Section 2 constructs the putative bounce for a process modeled on $p \to n\pi^+$. In Rindler coordinates the initial charged particle is static, whereas the neutral and charged daughters join it at two vertices. In Euclidean Cartesian coordinates the charged trajectories are circular arcs and the neutral trajectory is a straight segment. Solving the junction conditions gives the arbitrary-mass expression in equation (17). The paper then checks that expression against two limits quoted from Nikishov and Ritus: equal initial and final charged masses, equations (18)–(20), and a small charged-mass splitting, equations (21)–(25). In the light charged-daughter regime it obtains equation (26), discusses the real-bounce domain $m_c \geq m_n - m_p$, and analytically continues the result to $m_c < m_n - m_p$, where equation (35) asserts

$$\mathrm{Re}\, S_{\mathrm{inst}} = \frac{\pi m_c^2}{|eE|}.$$

Section 3 attempts an independent check from the imaginary part of the one-loop self-energy. The charged propagator is written in proper time, the Schwinger poles are summed, and the remaining integral is reduced to equation (47). In the weak-field limit, a saddle of the proper-time variable leads to equations (49)–(53). In the hierarchy $m_c \ll m_p, m_n$ and $m_n - m_p \ll m_c$, the resulting exponent agrees with the light-daughter expansion of the worldline action.

Sections 4 and 5 discuss consequences of this result. Section 4 interprets transverse recoil correlations as bipartite momentum entanglement and estimates a resolution-dependent entropy from an approximately Gaussian transverse distribution. Section 5 applies the electric-field construction to $p \to ne^+\nu_e$. It argues that the neutron and a massive neutrino share the same Euclidean line, so that the exponent reduces to the two-body answer with m_n replaced by $m_n + m_\nu$. The neutrino mass is then neglected and the complex-bounce result is used to obtain equation (68).

Section 6 treats decay in a magnetic field. The exact overlap of initial and final Landau wavefunctions is expressed in terms of associated Laguerre polynomials. A large-Landau-level saddle is claimed to give equations (85)–(92), followed by ultrarelativistic and near-threshold limits. A Routhian worldline construction is then used to reproduce the same exponent. The authors also compare a fixed-energy limit with the standard hard synchrotron tail and extend the analysis to photon-assisted decay. The manuscript closes with possible applications to proton conversion near charged black holes, entanglement, and holographic or chiral descriptions of baryon decay.

Overall assessment

The subject is interesting. Composite worldline saddles provide an economical geometric language for processes that are otherwise described using exact external-field wavefunctions, and a convincing equivalence among the worldline, self-energy, and Landau-overlap formulations would be useful. The arbitrary-mass electric-field action, a documented Rindler/Euclidean comparison, and the charged-particle magnetic Routhian construction could constitute publishable results. Several reductions of the electric action are internally consistent, and the Routhian identity in equations (110)–(114) captures the same geometric combination that appears in the Landau-overlap calculation.

In its present form, however, the manuscript does not yet establish its central claims at journal standard. The self-energy calculation omits the charged external-state projection that defines the decay rate, so it is not presently an independent derivation of the advertised quantity. The complex saddle used for physical weak proton decay is selected by analytic continuation without a contour, Stokes, or negative-mode argument. Most seriously, the magnetic analysis contains an incorrect Stirling term and several dimensionally inconsistent or wrong-sign formulas in the main results. The entanglement estimates also use more information than is controlled by a leading-exponential calculation. These issues affect correctness rather than presentation alone and require a substantial revision.

The authors should also sharpen the novelty claim. The paper reproduces limits already attributed to Nikishov–Ritus, standard Landau-level overlaps, the hard synchrotron exponent, and a crossed-field proton result. A revised version should state explicitly which formulas and equivalences are new, and should compare them systematically with the cited literature rather than treating recovery of known limits as sufficient evidence for all parameter regimes.

Major comments

1. **The self-energy calculation does not yet compute the decay rate defined in equation (36).**

   Equation (36) correctly identifies the relevant object as

   $$\langle p, \mathrm{in}| \operatorname{Im} \Sigma_E |p, \mathrm{in}\rangle.$$

   The manuscript then explicitly does not perform this projection. Instead it studies a momentum-space kernel and, in equation (139), analytically continues p_E^2 to $-m_p^2$. A charged particle in a constant electric field is not an ordinary plane-wave mass-shell state. The propagator is translationally invariant only after separating a gauge-dependent Schwinger phase, and the external charged state must be represented in an appropriate accelerated-state or Ritus basis. It is not automatic that projecting the kernel onto that state leaves the leading exponent unchanged.

   This omission matters because the manuscript repeatedly presents Section 3 as an independent validation of the composite instanton. The authors should either complete the gauge-covariant external-state calculation or prove, at leading exponential accuracy, that the omitted projection and phase factors are exponent-neutral. The assumptions on the preparation and normalization of the in-state should be stated. If that cannot be done, the abstract, Introduction, and Discussion should be narrowed: Section 3 would then provide a suggestive comparison of kernels in a restricted hierarchy, not a derivation of the physical decay rate.

Even within its present treatment, the agreement is more limited than the prose suggests. At $k_\perp = 0$, equation (51) gives

$$S_* = \frac{1}{|eE|}\left[\frac{\pi m_c^2}{2} + (m_n - m_p)\sqrt{m_c^2 - (m_n - m_p)^2} + m_c^2 \arcsin\left(\frac{m_n - m_p}{m_c}\right)\right],$$

which equals equation (26) by $\arccos(-x) = \pi/2 + \arcsin x$. This is a useful check, but it checks the light-daughter expansion, not the full arbitrary-mass result in equation (17). The revised manuscript should state this scope precisely.

2. **The complex-bounce regime central to Section 5 requires a saddle-contour and endpoint analysis.**

   Equations (30)–(35) choose branches of the inverse trigonometric functions and then take the real part of the continued action. The algebra on that branch is straightforward, but a complex classical solution contributes only if it lies on the relevant integration cycle. The paper does not identify the thimble, show how it is reached as m_c crosses $m_n - m_p$, determine the Stokes behavior, or establish the appropriate instability/negative-mode structure. Merely choosing the continuation with no extra winding is not a substitute for this analysis.

   The issue is especially important because physical beta decay lies on the other side of the real-bounce boundary: $m_n - m_p > m_e$. In the notation of Section 3 this means $\Delta^2 > \nu'$, whereas the interior saddle used in equations (51)–(53) exists only for $\Delta^2 < \nu'$. The authors' own exponent in equation (49) suggests a useful way to analyze the missing regime. When $\Delta^2 > \nu'$, the interior saddle disappears and the minimum moves to the endpoint $\tau = 0$, formally giving

$$S(0) = 2\pi\nu' = \frac{\pi(m_c^2 + k_\perp^2)}{|eE|}.$$

   A controlled endpoint asymptotic, including the validity of the Bessel approximation in its boundary layer and the proper contour prescription, could support equation (35). It should be carried out before the result is applied in Section 5.

   The reduction of the three-body process also deserves more care. For $m_\nu > 0$, two neutral worldlines connecting the same Euclidean vertices can share the same straight path and their classical length terms combine. The physical $m_\nu \to 0$ limit is nevertheless singular in the worldline representation, as the manuscript itself notes immediately before taking that limit. The authors should show that the desired exponent is the controlled $m_\nu \to 0^+$ limit, identify the soft-neutrino boundary saddle, and explain how the weak matrix element and phase space enter the prefactor. Without this, equation (68) is not adequately established.

3. **Equation (85) contains an error in the leading exponential obtained from the Landau overlap.**

   Starting from equation (80), set $n = A/b$, $l = B/b$, $n + l = Q/b$, and $K^2/2 = C$. Stirling's formula applied to

$$e^{-C/b}\frac{(A/b)!}{(Q/b)!}\left(\frac{C}{b}\right)^{B/b}$$

   and the saddle $L_{A/b}^{B/b}(C/b) \sim e^{f(z_*)/b}$ gives

$$S = C - B + A\log\frac{Q}{A} + B\log\frac{Q}{C} + 2A\log z_* + 2B\log(1 - z_*) + \frac{2Cz_*}{1 - z_*}.$$

The manuscript instead prints $B\log(Q/B)$. With the printed term, equation (85) does not reduce to equation (87). The problem is not cosmetic: for admissible numerical choices it can even make the putative S negative, which would predict an overlap exponentially larger than one. For example, $P_p = 2$, $P_c = 1$, and $K = 0.5$ give approximately -2.41 from the printed equation (85), while equation (87) gives a positive value of approximately 1.32.

The authors should correct this derivation and recheck every subsequent formula that depends on it. They should also justify the choice of z_- as the contributing steepest-descent saddle, specify the branch choices for the logarithms, and state the allowed domain of P_p, P_c, and K. Calling z_- simply a "minimum" of a complex contour exponent is not sufficient.

4. **The ultrarelativistic and photon-assisted magnetic exponents contain several mutually inconsistent errors.**

   Expanding equation (87) in the ultrarelativistic regime and using equation (95) gives

$$S^{\rm UR} = \frac{2\sqrt{3}}{|eH|E_p}\frac{1-x_*}{x_*}\left(m_c^2 - \frac{x_*^2 m_n^2}{(1-x_*)^2}\right)^{3/2}.$$

   Thus the factor M in equation (98) is undefined and extraneous. It also makes the expression dimensionful. In the hierarchy $m_p = M$, $m_n = M + \delta$, $m_c = \mu$, the reduced answer instead requires the factor M that is missing from equation (99):

$$S^{\rm UR} = \frac{\sqrt{3}\,M}{8|eH|E_p}\sqrt{s-\delta}\,(s+3\delta)^{3/2}, \qquad s = \sqrt{\delta^2 + 8\mu^2}.$$

   As printed, both equations (98) and (99) fail dimensional analysis and cannot support the stated crossed-field comparison.

   The induced-decay formulas expose the same inconsistency internally. At $\Omega = 0$, equation (125) reduces to equation (96), and equation (126) should reduce to equation (98); this occurs only after removing the extra M from equation (98). Equation (124) also has the wrong sign in the rate: the exponent must be negative. The printed positive sign predicts exponential growth as $H \to 0$. Finally, equation (128) contains $(s_\eta + 3\delta_\eta)^3$, which must be $(s_\eta + 3\delta_\eta)^{3/2}$ in order to be dimensionless and to reduce to the corrected equation (99) at $\eta = 0$. All magnetic limits and comparisons should be redone after these corrections.

   There is a further inconsistency in Section 6.5. With $x = (E-\omega)/E$, the standard synchrotron argument is

$$\xi = \frac{2(1-x)}{3\chi x},$$

   not $2(1-x)/(3\chi)$ as stated below equation (116). The missing $1/x$ is nevertheless used in the exponent printed in equation (118). The definitions and comparison should be made consistent.

5. **The magnetic saddle and Routhian equivalence need a reproducible derivation over a stated domain.**

   Several nontrivial steps in Section 6.1 are asserted rather than demonstrated: the contour deformation to z_-, conversion of the Landau-level sum to an integral, the claim that $k_1 = 0$ is the relevant saddle, and minimization with respect to P_c. Boundary saddles can compete when K approaches either endpoint of $0 < K < P_p - P_c$, and the range in which the large-N Laguerre asymptotic is uniform should be stated. A numerical comparison of the exact overlap in equation (80) with the corrected exponential would be valuable and would test the branch selection.

The Routhian construction in Section 6.4 is potentially one of the more interesting parts of the paper, but its variational problem should be specified more carefully. The authors should say which energy and canonical momentum are fixed at each endpoint, how the Legendre terms in equation (104) implement Landau-state rather than coordinate boundary conditions, and which contour makes the imaginary momentum trajectory contribute. They should also discuss the fluctuation/negative-mode criterion for interpreting the composite configuration as a decay saddle. The prose before equation (107) writes the tunneling constraint with the wrong sign for P^2, although equation (107) itself has the sign consistent with equations (108)–(109).

Equation (79) should also be corrected before the asymptotic analysis. The normal-ordered displacement operator is the product

$$e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a},$$

not the nested exponential that is presently printed.

6. **The entanglement entropy is not controlled by the leading exponent at the precision claimed.**

The paper obtains a normalized transverse distribution by retaining only the exponential part of the differential rate and then uses it to compute the full constant term in equation (63). A normalized entropy is sensitive to the prefactor that has been discarded. At best, the calculation can establish a leading logarithmic dependence in a clearly specified hierarchy and resolution window; it cannot establish the precise additive constant without the differential prefactor.

Moreover, equation (61) is not the general transverse dependence implied by equation (51). For $0 < \delta = m_n - m_p < m_c$, direct differentiation of the saddle exponent gives

$$\left.\frac{\partial S_*}{\partial k_\perp^2}\right|_{k_\perp=0} = \frac{1}{|eE|}\left[\frac{\pi}{2} + \arcsin\left(\frac{\delta}{m_c}\right) + \frac{\sqrt{m_c^2-\delta^2}}{m_n}\right].$$

The value $\pi/(2|eE|)$ in equation (61) therefore requires the additional double hierarchy $\delta/m_c \ll 1$ and $m_c/m_n \ll 1$. Those approximations should be stated and their effect on the entropy should be tracked.

The problem is sharper in Section 5. In the complex/endpoint regime, the formal exponent above is $\pi(m_e^2 + k_\perp^2)/|eE|$, not $\pi(m_e^2 + k_\perp^2)/(2|eE|)$. Copying equation (63) directly into equation (69) is therefore inconsistent even at exponential order and changes the answer by $\ln 2$. A revised treatment should distinguish the real-saddle and endpoint regimes, include the mass-dependent width, specify $(\Delta k)^2 \ll |eE|$ when a continuum Gaussian is used, and present only the entropy accuracy actually supported by the calculation.

7. **The claimed Rindler/Euclidean equality is not documented.**

Section 2.1 supplies the Rindler equations of motion and junction equations, but it does not write the complete on-shell Rindler action, explain the gauge and boundary terms under the coordinate change, or exhibit the numerical comparison. The statements that agreement is "exact within numerical precision" in Section 2.2 are unsupported by parameter choices, a table, a plot, an error measure, or reproducible code. Since this equality is presented as a check on the less direct Euclidean construction, it should be shown explicitly. An analytic mapping would be preferable; otherwise, a representative scan over the allowed mass domain with numerical errors should be included.

The domain of equation (17) should also be stated before the continuation is discussed. The real junction geometry requires the relevant triangle inequalities and a definite orientation of the charged arcs. The charge assignments and the sign of E should be fixed once, and all suppression exponents should be expressed using $|qE|$.

8. **Novelty, significance, and physical applicability should be reassessed after the corrections.**

   The manuscript should separate the arbitrary-mass electric bounce and the charged magnetic Routhian construction from results already contained in, or directly implied by, the cited work of Nikishov–Ritus, the earlier composite-instanton literature, the magnetic photon/neutrino calculations, standard synchrotron asymptotics, and the crossed-field proton-transmutation calculation. A concise table listing each claimed result, its domain, and whether it is new, a rederivation, or a consistency check would make the contribution much clearer.

   The black-hole discussion in Section 7 is currently too strong. The paper computes only a leading exponential in idealized constant fields, while the paragraph of interest concerns fields described as near critical, inhomogeneous geometry, and a comparison between weak decay and Schwinger production. When the two processes have comparable exponential factors, their relative importance is determined by the weak and electromagnetic prefactors, spin structure, phase space, and local geometry – precisely the ingredients not computed here. The application may be retained as motivation, but it should be presented as qualitative until those effects are evaluated.

Minor comments

1. The Hamiltonian in equation (70) omits the longitudinal term p_1^2, although p_1 appears in the wavefunctions and phase-space integral below. The intended restriction should be stated, or the term should be restored.

2. The notation should consistently distinguish the field magnitude from its signed component. Several electric formulas alternate between eE and $|eE|$, while the magnetic formulas assume $eH > 0$ without saying so.

3. Equation (59) has an extra closing parenthesis inside the logarithm, and equation (54) writes H_π rather than $\mathcal{H}_\pi$.

4. Equation (103) contains $\log(1/\varepsilon)$ with dimensionful ε. A reference energy is required, for example $\log(E_0/\varepsilon)$. The statement that P_c is "exponentially suppressed in ε" should be supported by the explicit asymptotic solution.

5. In the sentence following equation (124), the saddle relation is written with a lower-case k in the denominator, whereas the preceding equations use K.

6. In the Appendix, the polar decomposition used between equations (144) and (145) should read $\tau \pm i = \sqrt{1+\tau^2}\exp[\pm i \arctan(1/\tau)]$; the factor i is missing in the text.

7. The scalar interaction that defines the coupling g should be stated explicitly. This would fix normalization conventions and make clear which aspects of the prefactor and spinful weak vertex are intentionally omitted.

8. The statement that spin never influences the leading exponent should be qualified by the assumptions of minimal coupling and a nonvanishing saddle-point matrix element. Helicity or vertex zeros can

change which saddle or endpoint dominates even if they do not alter a generic classical worldline action.

9. The equal-mass and small-splitting comparisons in Section 2 would be easier to assess if the precise equations from the cited Nikishov–Ritus work and the mapping of notation were given.

10. The continuum entropy depends on the coarse graining. The physical meaning of $R_{\rm coh}$, the field duration and transverse area, and the normalization of continuum momentum states should be stated before equation (63).

11. A number of grammatical and typographical issues remain, including repeated words, missing spaces before references, "Lagger" for "Laguerre," and inconsistent use of "probability," "rate," and "decay exponent." A careful copy edit is needed after the technical revision.

12. The manuscript should use one notation for the charged daughter (m_π, m_c, or μ) within each derivation and announce every change. The current switches make the domains of approximations harder to follow.

Recommendation

The geometric idea is worthwhile and the electric-field action appears to contain a potentially useful core. I would be willing to reconsider a substantially revised manuscript. Before publication, however, the authors should complete or rigorously delimit the external-state self-energy check; establish the complex/endpoint saddle used for weak decay; correct equations (85), (98), (99), (124), and (128) and repeat the dependent magnetic analysis; recompute the transverse distribution at the accuracy claimed; and document the Rindler/Euclidean comparison. These revisions are necessary to determine whether the principal results are correct, novel, and quantitatively reliable.