Referee Report

"Liouville strings in AdS_3: the worldsheet story"

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Recommendation: Major revision.

Summary of the paper

The manuscript proposes an alternative representation of the bosonic $SL(2,\mathbb{R})_k$ worldsheet theory for strings on AdS_3 with NS–NS flux. The proposed field content consists of a spacelike Liouville field ϕ_+ with background charge $Q_+ = b + b^{-1}$, a timelike scalar ϕ_- with real background charge $Q_- = \sqrt{k}$, and a free scalar X, where $b^{-2} = k - 2$. In addition to the usual Liouville wall, the action is deformed by

$$\mathcal{O} = \exp\left[-\frac{\phi_+}{\sqrt{2}b} - \sqrt{\frac{k}{2}}\phi_-\right].$$

The total central charge agrees with $3k/(k-2)$. The momenta of exponential vertices are parametrized by $(j, m, \bar{m}, \omega)$ so that their weights take the standard spectrally flowed WZW form in equation (2.13).

Section 3 expands correlators in powers of $\mathcal{O}$ and integrates out X and ϕ_-. Charge neutrality relates the perturbative order ℓ to total spectral flow, and the remaining factors are Liouville correlators with ℓ insertions of the degenerate field $V^L_{-1/(2b)}$. After setting $r = n - 2 - \ell$, equations (3.13)–(3.16) are identified with the generalized H_3^+ WZW–Liouville relation for an n-point function with r units of winding violation. Section 4 then interprets this relation as an equality of string amplitudes and uses KPZ scaling to relate the Liouville cosmological constant μ to the spacetime string coupling.

Section 5 studies the deformation itself. The manuscript assigns it $(j, m, \bar{m}, \omega) = (1 - k, k, k, -2)$, verifies that it has weights $(1, 1)$ and zero spacetime energy, and attempts to relate it through spectral-flow duality and Weyl reflection to the familiar dimension-zero spectral-flow operator. It contrasts this $|\omega| = 2$ deformation with the $|\omega| = 1$ Liouville-wall deformation used in an earlier free-field description. The final part compares the construction with the Liouville-dressed twist-two deformation of the proposed spacetime symmetric-orbifold CFT. Appendices review the long-string stress-tensor improvement and the deformed symmetric product.

Overall assessment

The organizing idea is interesting. Several nontrivial algebraic checks work: equations (2.3)–(2.5) give the WZW central charge; equation (2.13) correctly reproduces the flowed conformal weights; the neutrality conditions (3.5)–(3.6) have the expected relation to winding violation; and the coordinate exponents in equations (3.9)–(3.10) and (3.14)–(3.15) expand correctly. The dimension-one check in equations (5.5)–(5.7) is also sound. Reinterpreting the degenerate insertion in the generalized WZW–Liouville formula as an action-level deformation could give a useful physical account of winding-changing amplitudes.

The central claims are nevertheless not established at journal standard in the present version. The timelike path integral and its zero modes do not have a stated contour prescription; exact marginality and preservation of the full affine algebra are asserted rather than shown; and the claimed finite truncation of the perturbation series does not follow from the Liouville argument given. In addition, direct substitution reveals errors in the KPZ exponent, the spectral-flow transformation, the radial interpretation of the deformation, and the long-string improvement. The one-loop formula is not a consistent higher-genus extension. These are issues of correctness and scope, not merely presentation.

The novelty should also be stated more precisely. The principal formula is explicitly the known generalized H_3^+–Liouville correspondence. The potentially new contribution is the claim that this formula arises from a well-defined deformed CFT, together with the $\omega = -2$ interpretation and the comparison with the spacetime twist-two operator. That would be significant if proved. At present, the manuscript provides a promising formal Coulomb-gas packaging of selected sphere correlators, but not yet an equivalence of complete worldsheet CFTs.

Major comments

1. **The proposed deformed theory needs a precise definition and an exact-marginality argument.**

   The ϕ_- kinetic term is timelike, while the X and ϕ_- vertices have real exponential coefficients before the double Wick rotation. A real zero-mode integral of the form

   $$\int dc \, \exp\left(\sqrt{2}c \sum_i p_i\right)$$

   does not literally produce the Dirac delta functions in equations (3.2)–(3.3); it diverges. The interaction also makes a naive real Euclidean integral unbounded in some directions. These features are standardly handled by analytic continuation in related constructions, but the manuscript must state the integration contours, the initial domain of momenta, the continuation to Lorentzian AdS_3, and the regulation of collisions and zero-mode volumes. The conversion of two continuous deltas into one Kronecker and one Dirac delta also requires the compactification and normalization conventions, including the Jacobian, if the final coefficient is to be called exact.

   Equation (5.6) proves that $\mathcal{O}$ is marginal to first order. It does not by itself prove the statement immediately following it that the operator is exactly marginal. The authors should analyze the $\mathcal{O}\times\mathcal{O}$ OPE and its mixed OPE with the Liouville wall, including contact terms of integrated insertions and possible generated operators. A screening-charge proof may be available, but it should be given. Since $\mathcal{O}$ is non-normalizable, the paper should also explain whether it is a modulus of one CFT or a deformation that changes asymptotic boundary conditions.

   Finally, only J^3 is displayed in equations (2.17)–(2.19). Equality of the central charge and of a family of conformal weights does not establish the $\widehat{sl}(2)_k$ algebra. The revised paper should exhibit $J^\pm$ and check their OPEs and conservation in the presence of both interactions, or give an equivalent all-orders argument. It should also specify the reflection relation, continuous and discrete sectors, and operator-product closure. Until then, the claims of an equivalent CFT should be replaced by the more limited correlator statement actually derived.

2. **The finite range of degenerate insertions after equation (3.11) is not demonstrated.**

   The manuscript says that the Liouville correlator "may vanish for $\ell \geq n-2$" and then retains $\ell = n-2$ in the sum. This is already internally contradictory: that endpoint is the $r = 0$ contribution and is needed in equation (3.13). More importantly, the cited special DOZZ zero does not prove a general vanishing theorem for the $(n+\ell)$-point correlator in equation (3.11). Liouville correlators with multiple degenerate fields can be nonzero when their fusion channels are compatible; the number of degenerate insertions alone does not give the asserted bound.

For fixed external charges, equations (3.5)–(3.6) select one value of ℓ. The standard restriction $0 \leq r \leq n-2$ is a WZW winding selection rule, and it also follows only after imposing a definite sign branch for the external spectral-flow labels. The authors should either prove a Liouville vanishing result for precisely the momenta and residue prescription at hand, or derive the allowed range from an independently stated WZW selection rule and stop presenting that rule as a consequence of a generic DOZZ zero. Using the WZW rule without acknowledgment would be circular if the same step is then used to prove WZW equivalence.

The domain of external states also needs attention. The assertion below equation (2.16) that one may take all $m, \bar{m} > 0$ and all $\omega < 0$ "without loss of generality" does not cover a generic amplitude with mixed incoming and outgoing vertices. A global time reversal does not make all labels in such an amplitude have the chosen sign. The revised paper should give explicit in/out and conjugation conventions and show how the opposite winding branch is represented by the action, possibly through a conjugate screening.

3. **The exact sphere-correlator match and its normalizations need a controlled statement.**

   Equation (3.12) contains $\Gamma(j_i - m_i)$, whereas the definition (2.11), the preceding derivation, and equation (3.13) require $\Gamma(-j_i - m_i)$. The recurrence stated below equation (2.11) is also off by one: the displayed gamma function gives

$$N_{j,m,\bar{m}} = (-j-m-1)N_{j,m+1,\bar{m}},$$

   not $(-j-m)N_{j,m+1,\bar{m}}$. These mistakes must be corrected before an exact normalization claim can be assessed. The phase identity under $m \leftrightarrow \bar{m}$ should be rechecked for the stated representation domain as well.

   The status of the maximally winding-violating endpoint $r = n-2$ should be discussed explicitly. In the cited generalized correspondence this sector, where no degenerate Liouville insertions remain, requires a special limiting argument and is not controlled by the same BPZ derivation as the generic case. Any undetermined k-dependent normalization inherited from that literature should be separated from what is proved here. A direct check of the two- and three-point functions, including reflection factors and delta normalizations, would substantially strengthen the proposal.

   For a fixed set of ω_i, the delta function in the formula following equation (3.16) selects at most one r; it is therefore misleading to say that such a fixed-label correlator is a physical sum of different total winding numbers. A genuine sum occurs only after a physical vertex is defined as a superposition over ω. Also, $r = 0, \ldots, n-2$ contains $n-1$ contributions, not $n-2$ as stated.

4. **The KPZ/string-coupling argument contains an uncancelled term.**

   From the first expression in equation (4.6) and equation (3.16),

$$\frac{\alpha_+^i}{b} = j_i + 1 + \frac{1}{2b^2},$$

   one obtains

$$\widehat{\kappa} = 1 + \frac{1}{b^2} - \sum_{i=1}^{n}(j_i+1),$$

not $1-\sum_i(j_i+1)$. Thus equation (4.6) drops b^{-2}. At genus g the corresponding missing term is $(1-g)b^{-2}$. As written, the claimed matching and the inference $\mu \sim g_s^{-2}$ do not follow.

There is a possible resolution that should be made explicit rather than hidden in the algebra. The universal sphere factor λ^{-2} in equation (3.13) scales as μ^{1/b^2} and was earlier described as an absorbable normalization of the inner product. Dividing by that factor would remove the discrepancy on the sphere. If this is intended, the authors must define the normalized observables before doing the KPZ count, explain the Euler-characteristic dependence at higher genus, and apply the same convention consistently to every amplitude. The invariant combination of λ and μ under a zero-mode shift should also be stated; equation (4.2) is a scaling relation, not by itself a dynamical identification of two independently normalized couplings.

5. **The spectral-flow transformation in equations (5.12)–(5.16) is algebraically inconsistent.**

   Substituting the displayed map

$$\widetilde{j} = -\frac{k}{2} - j, \qquad \widetilde{m} = m + \frac{k}{2}, \qquad \widetilde{\omega} = \omega - 1$$

   into equation (2.13) gives

$$\widetilde{\Delta} - \Delta = m - j,$$

   not $-j-m$ as claimed in equation (5.13). The latter result follows from the inverse choice $\widetilde{m} = m - k/2$ and $\widetilde{\omega} = \omega + 1$. The invariance condition and the labels in equation (5.15) consequently do not follow from equation (5.12).

   This error affects a central physical interpretation: the relation between the dimension-one $(1-k, k, -2)$ operator and the standard dimension-zero spectral-flow operator is used to distinguish this construction from the $\omega = -1$ deformation of Ref. [28]. The authors should rederive the representation isomorphism with a single consistent convention, including the Weyl reflection and the highest/lowest-weight labels. If the dimension-one field is an affine descendant of the dimension-zero spectral-flow operator, that descendant and its normalization should be shown explicitly rather than inferred from the presently inconsistent map.

6. **The claimed deep-bulk localization is reversed by the manuscript's own field identification.**

   Section 5.1 states that the Wakimoto boundary is at $\varphi \to +\infty$ and identifies $\phi_+ = -\varphi$. It then follows that

$$\mathcal{O} \sim e^{-\sqrt{(k-2)/2}\,\phi_+} = e^{+\sqrt{(k-2)/2}\,\varphi}$$

   grows toward the AdS boundary. Conversely, $\mathcal{O}_L \sim e^{+\sqrt{2/(k-2)}\phi_+}$ is suppressed there. This is opposite to the repeated statements in Sections 5.1, 5.2, and the Conclusions that $\mathcal{O}$ acts deep in the bulk.

   The radial convention or the physical interpretation must be corrected. This matters because localization of winding-number violation is one of the paper's main qualitative claims. The phrases "tachyonic shock wave" and "suppressed at large k" should likewise be tied to a specified classical profile and scaling of fields and couplings; the exponent alone does not establish either conclusion.

7. **The higher-genus discussion is not a derived or internally consistent extension.**

   In equation (4.12) the measure integrates n variables while the integrand and Liouville correlator contain only $n-r$ insertion points. The measure should presumably be $\prod_{a=1}^{n-r} d^2y_a$. Moreover, the genus-one neutrality relation gives a perturbative power $\lambda^{\ell-n} = \lambda^{-r}$ when $\ell = n - r$, not the sphere power λ^{-r-2} printed in equation (4.12). The range $r = 0, \dots, n$ contains $n+1$ terms, not n.

A torus free-field calculation must also treat harmonic zero modes and loop momenta, determinants, the chiral phases suppressed by the use of $\chi(z|\tau)$, modular covariance, ghosts, and integration over the torus modulus. None of these is derived. The paper should either supply a complete genus-one calculation and a consistent genus-g normalization, or label Section 4.3 as conjectural and remove it as evidence for the claimed CFT equivalence.

8. **The charge-neutrality comparison with the orbifold deformation has an unresolved sign mismatch.**

   At $\mu = 0$, neutrality of the Liouville correlator in equation (3.11) requires

$$\sum_i b\left(j_i + \frac{k}{2}\right) - \frac{\ell}{2b} = b + \frac{1}{b}.$$

   With the manuscript's definition $r = n - 2 - \ell$, this gives

$$\sum_i j_i = 1 - n - \frac{k-2}{2} r.$$

   By contrast, equation (B.12), with the same displayed definition of r, gives

$$\sum_i j_i = 1 - n + \frac{k-2}{2} r.$$

   The comment about opposite ω conventions may be intended to identify $r_{\text{orb}} = -r_{\text{WS}}$, but that separate variable and the accompanying map of j conventions are never written. The asserted exact match should be replaced by an explicit convention-by-convention derivation. The discussion after equation (B.13) must also distinguish the degree of a cover, the number of covering maps, and the number of Liouville screenings; these are not the same quantity.

9. **Several elementary identities used in the physical dictionary and Appendix A must be corrected and their consequences traced.**

   Equation (2.12) is off by a factor of two. Equations (2.10) imply

$$\frac{\alpha_- + p}{\sqrt{2}} = \sqrt{\frac{k}{8}}(1+\omega),$$

   not $\sqrt{k/2}(1+\omega)$. Following equation (2.23), if $p_0 = p$ then $p_0 = (m + k\omega/2)/\sqrt{k}$; the quantity $m + k\omega/2$ is $\sqrt{k}p_0$, the J_0^3 charge. These normalizations should be fixed before the target-space interpretation is used.

   There is a more consequential sign problem in Appendix A. Equation (2.19) gives

$$\partial J^3(z) V(0) = -\frac{m + k\omega/2}{z^2} V(0) + \cdots .$$

   Therefore the improvement $\widehat{T} = T + \partial J^3$ shifts the weight by $-(m + k\omega/2)$, not by the positive charge printed in equation (A.11). Furthermore, $\partial^2 X(z) V(0)$ has no independent $z^{-1}\partial V$ pole, contrary to equation (A.12). The sign of the improvement, the statement that J^+ acquires weight zero, and the long-string weights must be made mutually consistent.

   Finally, equations (A.9)–(A.10) twice define $b = 1/(k-2)$, whereas all displayed background charges require $b = 1/\sqrt{k-2}$. Although the latter is likely a typographical error, the stress-tensor sign is not, and the appendix should be rederived rather than patched locally.

Minor comments

1. Equation (2.14) divides by ω and therefore does not cover the unflowed sector. The $\omega = 0$ physical-state condition should be stated separately.

2. The Wick-rotated vertex in equation (2.23) omits the radial ϕ-dependent factor. If this is intentional because only the (t, θ) dependence is being displayed, say so explicitly.

3. The abstract calls the deformation an $\omega = 2$ operator, while Section 5 uses $\omega = -2$. The paper should use $|\omega| = 2$ or announce the time-reversed convention.

4. The sentence before equation (5.4) refers to $\mathcal{O}_L$ and equation (5.2), but the listed momenta are those of $\mathcal{O}$ in equation (5.1).

5. The label used for $\bar{\beta}_{ij}$ is repeated for equations (3.10), (3.14), and (3.15), so cross-references are ambiguous. Unique labels should be assigned.

6. The manuscript should distinguish the Liouville coupling λ from the coupling denoted by the same letter in the symmetric-orbifold discussion unless their precise normalization relation is established.

7. The statement $c_+ \simeq 6k$ concerns one worldsheet factor whose leading contribution is canceled by the timelike sector in the total worldsheet central charge. It is not by itself a dynamical derivation of the full boundary CFT central charge. The comparison should be described as suggestive and tied to the single-long-string or seed central charge.

8. In Appendix B, $\hat{h}_2$ after equation (B.7) presumably means $\bar{h}_2$. The number N is first the number of covering sheets and then is called the number of Liouville screenings, while s is used for the latter in equation (B.14). These quantities should be kept distinct.

9. The reference to the paragraph "between equations (3.12) and (3.12)" in Appendix B is broken.

10. The paper would benefit from a concise table distinguishing results inherited from the H_3^+–Liouville correspondence, results inherited from the earlier $\omega = -1$ marginal-deformation construction, and new claims specific to the present $\omega = -2$ action.

11. A careful copy edit is needed. Examples include "linear dilaton" rather than "lineal dilaton," "internal manifold" rather than "integral manifold," "automorphism" rather than "authomorphism," and the phrase "not to vanish limit in the limit."

Novelty and significance

The paper addresses a technically mature but important corner of $\mathrm{AdS}_3/\mathrm{CFT}_2$. A genuine action-level realization of the generalized WZW–Liouville correspondence, with a precise explanation of winding violation, would be of interest to specialists in worldsheet CFT and stringy holography. The shared degenerate momentum $-1/(2b)$ in the worldsheet deformation and the spacetime twist-two deformation is an intriguing observation.

The manuscript should not, however, treat that shared momentum as evidence of a direct worldsheet/spacetime map. Appendix B establishes a parallel in charge-neutrality conditions, not an operator identification. The revised paper should compare its action explicitly with Refs. [33], [37], [28], and [36], state which correlation functions or structural results are new, and identify a prediction that does not simply restate the known generalized correspondence. If the authors can define the CFT, repair the spectral-flow map, and derive the sphere and torus observables with controlled normalizations, the conceptual contribution could be substantial.

Recommendation

I would be willing to reconsider a substantially revised manuscript. At a minimum, the authors should define the analytic continuation and zero-mode prescription; establish exact marginality and the full affine symmetry; replace the unsupported Liouville truncation with a valid derivation; correct the KPZ, spectral-flow, radial, and long-string arguments; and either derive or clearly withdraw the higher-genus claims. These revisions are necessary to determine whether the proposed action is a new equivalent worldsheet CFT or a useful but formal rewriting of known sphere correlators.