Referee Report

Compact, long-range and vacuumless regimes controlled by scalar fields in two-dimensional flat and five-dimensional warped geometries

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Recommendation: Major revision.

Summary and overall assessment

The manuscript studies three one-parameter families of canonical scalar-field models. Model A, defined in equation (12), has vacua at $\phi = \pm 1$ and a vacuum mass that grows with the integer n; its finite-n kink approaches a compacton only in the limit $n \to \infty$. Model B, equation (20), has massless vacua for $n > 1$ and consequently supports polynomially decaying kink tails. Model C, equation (30), moves the vacua to $\phi = \pm n$, so that the fixed-coordinate limit has $\phi(x) \to x$, a constant energy-density plateau, and divergent total energy. Section 4 applies the square-root deformation in equation (39). At its strict $\alpha \to \infty$ endpoint this produces the modulus potentials (43), (55), and (61). The first and third support compactons for every finite n, whereas the second is compact only for $n = 1$, has an exponential tail for $n = 2$, and has a polynomial tail for $n > 2$.

Section 5 embeds the associated first-order systems in a five-dimensional Einstein-scalar construction. The authors derive analytic or semi-analytic warp functions for the three undeformed families and for the modulus A and C families. The main claimed braneworld results are a long-range thick brane, a Gaussian wall in the Model-C vacuumless limit, and two families of finite-n hybrid branes with an exactly compact scalar core and an exponential exterior. The analytic catalogue is useful, and the arbitrary-n extension of the modulus compacton/hybrid construction is potentially publishable.

The principal first-order reductions, kink energies, implicit hypergeometric solutions, and flat-space tail exponents are mostly mutually consistent. Nevertheless, several issues affect the central physical conclusions rather than merely their presentation. The gravitational tail of Model B is exponential, contrary to the discussion following equation (85). The Gaussian geometry in equation (88) has divergent curvature invariants and is not asymptotically AdS_5, contrary to the claimed regularization of the vacuumless limit. Exact compactness occurs only at the singular endpoint $\alpha = \infty$, not at finite large α, and the stability operator for the resulting cusped potentials has not been defined. In addition, Model C is largely a rescaling of Model A, which substantially narrows the independent novelty. These points require a major revision before the work can be judged suitable for publication.

Correctness and logical structure

The organization from the standard BPS formalism in Section 2 to the three flat-space examples in Section 3 is clear. Equations (13)–(19) correctly exhibit the exponentially localized finite-n Model-A kink and its compact $n \to \infty$ limit. Equations (22)–(29) consistently connect the vanishing vacuum mass of Model B to its power-law scalar and energy-density tails. The energy relation $E_C = nE_A$, the implicit Model-C solution, and the fixed-x large-n plateau in equations (34)–(38) are also internally consistent. Likewise, the beta-function convergence test after equation (57) correctly separates the compact, exponential, and polynomial cases of the modulus-B family.

The logical problems arise when singular limits and gravitational asymptotics are interpreted. The large-α expansion (41) is an expansion at fixed positive V; it is not uniform at a vacuum, where $V = 0$. Consequently, the exact finite-α theory and the modulus endpoint have different tail types. In Section 5, a polynomial approach of the scalar to its vacuum is incorrectly transferred to the warp factor even though the superpotential tends to a nonzero constant. Conversely, exponential or Gaussian decay of a warp factor is treated as sufficient evidence for a regular localized-gravity background, without examining curvature, the graviton zero mode, or the fluctuation spectrum. The detailed comments below identify the revisions needed to repair these steps.

Novelty and significance

The manuscript itself attributes the Model-A construction, the square-root deformation, the $n = 1$ sine compacton, and the original hybrid-brane mechanism to earlier work, including the cited papers by Bazeia and collaborators. The long-range mechanism associated with higher-order massless vacua is also standard in the literature cited in Section 3.2. The clearest new contribution is therefore the arbitrary finite-n modulus extension in equations (43)–(48) and its piecewise warped counterpart beginning with equation (89). The compactness threshold for the modulus-B family and the analytic expression (85) are useful secondary results.

At present, however, the new warped solutions are characterized only by static profiles. No curvature regularity test, effective four-dimensional Planck scale, tensor or scalar stability analysis, Kaluza–Klein spectrum, or other observable is supplied. Moreover, the second advertised A/C hybrid family is generated by a direct scaling relation. Thus the physical significance and the degree of independent novelty are more modest than the Introduction and conclusion suggest. A revised manuscript could become substantially stronger by isolating the genuinely new formulas, comparing them directly with the specific earlier compacton and hybrid-wall constructions, and establishing at least the tensor-sector consistency of the proposed backgrounds.

Major comments

1. **The long-range scalar does not give the claimed long-range gravitational tail.**
   For the Model-B wall, equation (83) gives
   $$W(1) = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2\,\Gamma(n+3/2)} > 0$$
   for every finite n. Combining this with equation (72) shows that, on the right-hand side of the wall,
   $$A'(y) \longrightarrow -\frac{2}{3}W(1), \qquad e^{2A(y)} \sim \exp\left[-\frac{4W(1)}{3}|y|\right].$$
   Equation (75) then gives $\rho(y) \sim -(4/3)W(1)^2 e^{2A(y)}$. Both the warp factor and the brane energy density therefore have exponential leading tails. It is only the scalar field, and subleading corrections to A', that approach their asymptotic values by a power law. The statements following equation (85) and the interpretation of Figure 19 must be corrected accordingly.

There is still a meaningful large-n effect: since $W(1) \sim \sqrt{\pi}/(2\sqrt{n})$, the exponential localization length grows like $\sqrt{n}$. This makes localization nonuniform in n. In particular, the integral controlling the four-dimensional Planck scale grows without bound in the strict $n \to \infty$ limit unless another scale is retuned. The authors should derive the full asymptotic expansion, state the order of limits, and replace the power-law gravity claim with this physically different conclusion.

2. **The Gaussian Model-C limit is curvature-singular and is not asymptotically AdS_5.**

   Equation (88) gives $A(y) = -y^2/3$. For the metric (67), the Ricci scalar contains the combination $8A'' + 20(A')^2$, up to the overall Riemann-tensor sign convention. Its magnitude therefore grows as $80y^2/9$. More decisively, the Kretschmann invariant behaves as

   $$R_{MNPQ}R^{MNPQ} = 24(A')^4 + 16\left(A'' + (A')^2\right)^2 \sim \frac{640}{81}y^4.$$

   Thus the Gaussian falloff of e^{2A} does not regularize the geometry. The limit is not asymptotically AdS, and the curvature becomes unbounded at both ends of the extra dimension. The assertion in the Introduction and below equation (88) that the warped geometry regularizes the vacuumless divergence, or leaves the gravitational sector unaffected by it, is not tenable as stated.

   The authors should compute the curvature invariants for finite n, analyze the nonuniform $n \to \infty$ limit, and determine whether the limiting singularities are geodesically accessible. If the Gaussian background is to be retained, its singular status and the physical criterion under which it is acceptable must be stated explicitly. A finite zero-mode normalization by itself would not establish regularity.

3. **Exact compactness occurs only at the singular endpoint $\alpha = \infty$, not at finite $\alpha \gg 1$.**

   Near a vacuum, the exact deformation (39) has

   $$V_\alpha = (1 + \alpha/2)V + O(V^2)$$

   for every finite positive α. Hence, for Models A and C, the vacuum remains quadratic and a kink retains an exponential tail at every finite α. It never reaches the vacuum at a finite coordinate. The modulus potentials (43) and (61), their compact edges, and the corresponding exact thin-brane exteriors arise only after taking $\alpha \to \infty$ before taking the field to its vacuum. The expansion (41) is nonuniform at $V = 0$, as is already evident because its displayed correction does not reproduce the exact identity $V_\alpha(0) = 0$.

   Sections 4 and 5.2 should therefore be described consistently as strict endpoint theories, rather than as generic $\alpha \gg 1$ regimes. The authors should discuss the boundary layer and the noncommuting limits $\alpha \to \infty$ and $|x|, |y| \to \infty$. Ideally, at least one finite- α sequence should be shown to demonstrate how an exponentially tailed wall approaches the compact limiting profile. The allowed parameter domain also needs to be specified: calling α an arbitrary real parameter is incorrect globally, since negative α can make the radicand negative or reverse the vacuum curvature. The analysis appears to require $\alpha \geq 0$, with $\alpha = 0$ defined by continuity.

4. **Linear stability for the modulus compactons has not been defined.**

   The fluctuation formula (10) assumes that V is twice differentiable along the background. The modulus potentials (43), (55), and (61) are instead cusped at their vacua. Their first derivative is discontinuous there, so $V_{\phi\phi}$ is distributional or undefined at a compact edge. It is not enough to plot an "infinite wall" in Figures 12, 15, and 17. For example, in the interior of the $n = 1$ modulus-A compacton the ordinary second derivative is simply $U = -1$; the boundary behavior depends on how the cusp and its limiting fluctuation operator are defined.

The authors should either regularize with finite α and take a controlled operator limit, or formulate the second variation directly for the nonsmooth theory. In either approach they must derive the matching or boundary conditions at $|x| = L_A, L_C$, specify the self-adjoint domain, and show that the claimed spectrum is nonnegative. The same issue reappears at compact scalar edges in the warped solutions and must be incorporated into their perturbation analysis.

5. **Warp-factor plots do not establish localization or stability of four-dimensional gravity.**

   Section 5 infers localized gravity from the shape of e^{2A} and ρ, but a publishable braneworld claim requires more. At minimum, the authors should evaluate $M_{\text{Pl},4}^2 \propto \int dy\, e^{2A(y)}$, derive the transverse-traceless tensor perturbation equation, verify the normalizability of the graviton zero mode, and establish the absence of tensor tachyons with the correct edge conditions. The continuum threshold or resonant structure should be discussed where it changes qualitatively. Because the scalar potentials are nonsmooth in the hybrid cases, the scalar-gravity sector also deserves at least a clear statement of its junction conditions and possible singular terms.

   This calculation is especially important for Model B, whose localization length diverges with n, and for the Gaussian Model-C limit, whose Planck integral may be finite even though the geometry is curvature-singular. These examples show why decay of the plotted warp factor is not by itself the needed physical test.

6. **Model C is a scaling of Model A rather than an independent mechanism.**

   For the same integer n, let $\psi = \phi/n$ and $\xi = x/n$. The first-order equations imply

   $$\phi_C(x) = n\,\phi_A(x/n), \qquad E_C = nE_A.$$

   At the modulus endpoint one similarly has $L_C = nL_A$ and $E_\infty^C = nE_\infty^A$, as the manuscript partly notes. In the brane construction,

   $$W_C(\phi) = nW_A(\phi/n), \qquad A_C(y) = n^2 A_A(y/n)$$

   after imposing the same central normalization. Thus the second A/C family has no independent functional content until a fixed dimensional scale or coupling is specified that makes this rescaling physically nontrivial.

   The manuscript should make this equivalence explicit and revise the novelty claims in Sections 3.3, 4.3, and 5.2. If the authors regard Models A and C as physically distinct, they should identify which quantities are held fixed and exhibit a dimensionless observable that distinguishes the two constructions. The algebraic mapping to a beta-Starobinsky potential does not, by itself, provide such an observable in the soliton or braneworld problem studied here.

7. **The "vacuumless" limit and the relevant orders of limits need more precise physical interpretation.**

   At fixed finite ϕ, equation (30) tends to $V_C \to 1/2$ as $n \to \infty$; at fixed x, equation (38) gives $\phi(x) \to x$, $\rho(x) \to 1$, and an infinite total energy. The limiting object is not a localized finite-energy defect, and it differs from familiar vacuumless potentials that tend to zero only at infinite field. The vacua here recede to infinity while the pointwise potential becomes a constant. This distinction should be stated clearly when comparing with the vacuumless literature cited in Section 3.3.

The five-dimensional limit is also nonuniform. Every finite-n wall reaches an asymptotically AdS region whose curvature scale is set by $W(n)$, while the fixed-y limit gives the singular Gaussian profile (88). Taking $n \to \infty$ before $|y| \to \infty$ is therefore not equivalent to taking the asymptotic region first. The paper should track the moving transition scale, state which limiting geometry is meant, and avoid describing the divergent flat-space defect as regularized without these qualifications.

8. **The novelty claim needs a more discriminating comparison with the cited literature and a physical payoff.**

   The paper correctly cites prior work for compactons, the deformation (39), hybrid branes, long-range kinks, and the authors' earlier Model-C-type constructions. It should now state, result by result, what equations (43)–(48), (55)–(59), and (89)–(103) add to those predecessors. In particular, the arbitrary-n modulus-A hybrid appears to be the main independent extension; the modulus-B classification is elementary, and the modulus-C result is related to A by scaling. The broad statement that two new finite-n families have not been addressed in the literature is not sufficiently substantiated.

   The revision should also compare with later compacton/cusp-potential stability work, long-range thick branes, and Gaussian or soft-wall geometries, especially where curvature singularities and graviton localization have already been analyzed. Adding the perturbative gravity calculation requested above would turn the new profiles from a formal catalogue into a result of clearer interest to the high-energy theory community.

Minor comments

1. The text following equation (17) says that larger finite n changes the Model-A kink into a compacton with finite support. Equation (13), the later asymptotic discussion, and Table 1 show instead that every finite n has an exponential tail and infinite support. The phrase should be "approaches a compacton as $n \to \infty$."

2. The incomplete-beta representation stated after equation (44) is not odd under $\phi \to -\phi$ as written. With the standard real incomplete beta function it requires an overall $\mathrm{sgn}(\phi)$.

3. Table 1 lists the modulus-B support only for $n \geq 2$. It should also record explicitly that $n = 1$ has support $[-\pi/2, \pi/2]$.

4. Equation (86) defines a superpotential W, although its internal label and the caption of Figure 20 refer to it as a potential. The notation should distinguish throughout between the flat-space potential $V_{\mathrm{flat}} = W_\phi^2/2$ and the five-dimensional potential $V_5 = W_\phi^2/2 - 4W^2/3$.

5. The normalization of the warp function used in the plots, apparently $A(0) = 0$, should be stated. Since the asymptotic AdS scale changes with n, comparisons of widths should also say which dimensional quantity is being held fixed.

6. Near the discussion of equation (99), the energy density is written as $\rho(x)$; the five-dimensional coordinate there is y. The same coordinate convention should be checked throughout Section 5.

7. The statement that L_C increases linearly with n is not exact: $L_C = nL_A$, and L_A itself depends on n. It is asymptotically linear.

8. Equation (87) is typeset with a size command inside mathematics, which makes parts of the displayed expression anomalously small. This should be reformatted as an ordinary readable display.

9. The mode expansion preceding equation (9) uses only an integer index. Models with stability potentials tending to zero have a continuum above the threshold, so the spectral decomposition should include continuum modes and their normalization rather than only a discrete sum.

10. Section 5 imports the dimensionless flat-space superpotentials into the five-dimensional action without displaying the mass and length rescalings. The five-dimensional dimensions of ϕ, W, V_5, and y, and the relation of the suppressed scale to G_5, should be stated. This is also needed for meaningful comparisons of the n-dependent AdS radii.

11. The abstract is too general to communicate the actual result. It should identify which families compactify, state that exact compactness occurs at the strict modulus endpoint, and summarize the corrected brane asymptotics and regularity status.

Recommendation

The manuscript contains a coherent collection of analytic scalar profiles and a potentially useful finite-n extension of a known hybrid-brane mechanism. However, the incorrect gravitational asymptotics of Model B, the curvature singularity of the Gaussian limit, the nonuniform $\alpha \to \infty$ limit, and the undefined stability problem for cusped potentials presently prevent a positive publication recommendation. The Model-A/Model-C scaling relation and the absence of a graviton-sector analysis also weaken the novelty and physical significance. I recommend major revision. Reconsideration would be appropriate after the central five-dimensional claims are corrected, the limiting procedures and stability operators are made precise, and the new backgrounds are shown to support a consistent localized-gravity sector.