Referee Report

Non-Abelian A_4 vortices in $SO(3)$ gauge theory and non-invertible symmetries

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Summary and overall assessment

This manuscript constructs classical string solutions in a four-dimensional, perturbatively renormalizable $SO(3)$ gauge theory with a real spin-3 Higgs field. The Higgs potential is built from the quadratic invariant I_2 and a nonnegative quartic invariant Q. The proposed vacuum is proportional to the harmonic cubic $e_7 \propto XYZ$, whose stabilizer is the tetrahedral group A_4. On this basis the authors identify the vacuum manifold as $SO(3)/A_4$ and its fundamental group as the binary tetrahedral group $\widetilde{A}_4$.

The paper then gives two cylindrically symmetric families. Equations (33)–(39) define an order-two family in which the gauge field lies along T^3 and the Higgs field remains in the two-dimensional span of e_6, e_7. This is an embedded Abelian-Higgs system. Equations (44)–(52) define an order-three family with one gauge and two scalar profiles. The reduced field equations, mass normalizations, and tension functionals are largely internally consistent. In particular, equation (65) gives the expected Abelian-Higgs functional, equation (70) follows from the order-three ansatz, and the argument in Appendix C correctly implies $b(R) = 1$ at $\eta = 1$. The resulting relation $E_3 = (4/9)E_2$ in equations (72)–(75) is also correct within that reduced sector.

Section V usefully separates the projected A_4 holonomy from the full homotopy class in $\widetilde{A}_4$. Apart from an orientation error in the order-three assignment discussed below, the binary-tetrahedral conjugacy classes, their sizes, and the projection displayed in equations (80) and (81) are correct. Section VI reviews conjugacy-class flux operators in a pure A_4 gauge theory. With the unnormalized definition in equation (92), Table II is the correct A_4 class-sum algebra.

The basic construction is interesting. The synthesis of a simple $SO(3) \to A_4$ Higgs model, explicit smooth radial representatives, their reduced tensions, and a proposed relation to finite-group surface defects could be useful to the soliton and generalized-symmetry communities. The credible novelty is the concrete gauge-Higgs realization and its explicit profiles, rather than the already established binary-tetrahedral topology, non-Abelian Aharonov–Bohm phenomena, or finite-group class algebra.

Several headline conclusions, however, are not yet established. There is a definite sign error in the order-three holonomy. More fundamentally, the computed tensions are those of restricted embedded configurations, with no stability or minimality analysis in the full spin-3 theory. The type-I/type-II interpretation treats the ansatz labels as additive integer charges even though Section V shows that the physical topology is periodic. Generic order-three binding energies are not computed away from $\eta = 1$. Finally, equality of a projected holonomy does not by itself prove that a finite-tension dynamical vortex flows to the gauge-averaged topological operator of equation (92), and the advertised fusion and Aharonov–Bohm matching is not actually carried out. These issues affect correctness and the central physical interpretation, but appear repairable.

Recommendation

Major revision. The explicit classical construction is sufficiently novel and interesting to merit further consideration. Before publication, the authors should correct the order-three charge assignment, establish the vacuum orbit used in the topology, and distinguish exact embedded solutions from stable minimum-tension vortices. The claims about forces and BPS behavior should be restricted to what the reduced functionals show unless a full fluctuation and interaction analysis is supplied. The relation to non-invertible surface operators also requires an operator-level matching, not only a map of holonomies. With these revisions, the paper could become suitable for a journal such as PRD or JHEP.

Major comments

1. **The order-three holonomy in equation (49) has the opposite orientation.**

   Using the generators displayed in equation (4), one obtains

   $$A := i(T^1 + T^2 + T^3) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

   The exponent in equation (49) for $M = 1$ is $2\pi A/(3\sqrt{3})$. Direct exponentiation gives

   $$\exp\left(\frac{2\pi}{3\sqrt{3}}A\right) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = r_3^{-1} = r_3^2,$$

   where r_3 is the matrix defined in equation (23). It is not the matrix printed on the right-hand side of equation (49). Thus, with the present conventions, the order-three ansatz has $U_M = r_3^{-M}$ in equation (82), and its lift is represented by $\widetilde{r}_3^{-M}$.

   This changes the order-three labels in Table I. Keeping the definitions of C_i and D_i in equations (77) and (80), the rows for $M = 1, 2, 3, 4, 5$ should read, respectively,

   $$(C_2, D_3), \quad (C_3, D_5), \quad (C_0, D_1), \quad (C_2, D_4), \quad (C_3, D_2).$$

   The radial profiles and tensions are insensitive to this orientation reversal, but the topological and defect identifications are not. An equally simple repair would be to redefine the generator called r_3 in equation (23) to be its inverse, or to reverse the sign in the gauge-field ansatz, and then propagate that convention consistently. The authors should also check every later occurrence of C_2 versus C_3, particularly the example following equation (93).

2. **The equality locus of the Higgs potential must be shown to be the claimed vacuum orbit.**

   Equations (17)–(20) and the sum-of-squares expression in Appendix B establish $Q(\phi) \geq 0$ and show that ve_7 is a vacuum. They do not establish the stronger statement needed in equations (25) and (26): at fixed $I_2 = v^2$, every zero of Q must lie on one $SO(3)$ orbit, and the stabilizer of that orbit must be exactly A_4. Nonnegativity alone does not exclude an additional orbit, a disconnected component, or a lower-symmetry stratum in the equality locus.

   This point is foundational because the classification in equation (76) assumes that the entire vacuum manifold is $SO(3)/A_4$, not merely that one selected vacuum has an A_4 stabilizer. The authors should solve the equality conditions in the six squares of Appendix B, or invoke a precise invariant-theory result characterizing the minimizers of I_4/I_2^2 in the spin-3 representation. The argument should show both transitivity of the $SO(3)$ action on the normalized zero locus and the exact stabilizer. A short direct verification that the stabilizer of XYZ consists of the twelve proper tetrahedral rotations would complete the second part. Once this is supplied, equations (25), (26), and (76) will have the required basis.

3. **The type-I/type-II and BPS-like conclusions do not yet describe full-theory vortex energetics.**

   The order-two truncation is a useful exact embedded Abelian-Higgs sector, and equation (68) is the correct Bogomol'nyi result for its reduced functional. It does not follow that $E_2(N;\kappa)$ is the minimum tension in the corresponding sector of the full spin-3 theory. The ansatz retains only W^3 and $\phi \in \mathrm{span}\{e_6, e_7\}$. Transverse gauge and Higgs fluctuations, additional cylindrically symmetric core condensates, and non-axisymmetric modes are not tested. This is especially important because the full model contains the second Higgs mass m_- in equation (29), while the entire order-two solution and Figure 4 are independent of $\eta = m_+/m_-$. The unexamined modes can therefore change stability as η is varied even though the plotted tension does not.

   There is also a direct conflict between treating N as an Abelian winding number in Figure 4 and the actual topology in Table I. The $N = 1$ and $N = 3$ configurations lie in the same conjugacy class D_6, and $N = 4$ lies in the trivial class D_0. At $\kappa = 1$, equation (68) assigns energy 4 to the $N = 4$ embedded solution even though the vacuum in the same topological sector has zero energy. Likewise, the $N = 3$ solution cannot be regarded as the protected minimum of a charge-three sector, because no such additive integer sector exists. The comparison $E_2(N)$ versus $NE_2(1)$ is a constrained comparison between particular co-oriented embedded configurations, not a classification of all physical binding or decay channels.

   The same issue applies to M modulo six in the order-three family. Equations (68) and (75) are therefore Bogomol'nyi equalities only in reduced Abelian subsectors; they are not global topological bounds in a theory with finite $\widetilde{A}_4$ charge. The statement in Section VII that static multi-vortex configurations follow from the reduced form should either be supported by explicitly embedding the general Abelian-Higgs multi-center solutions and checking the full field equations, or be weakened. Even if such stationary multi-center embeddings exist, their stability against transverse modes is a separate question.

   For the present claims to stand, the paper needs at least a quadratic fluctuation analysis around the $N = 1, 2$ and $M = 1, 2$ solutions, including the additional spin-3 and gauge channels and their dependence on (η, κ). Ideally the authors should enlarge the cylindrical ansatz and determine the minimum tension in each $\widetilde{A}_4$ conjugacy class. If this is beyond scope, the abstract, Introduction, Section IV, and conclusion should consistently call the results embedded stationary solutions and reduced-sector binding energies. The words "attractive," "repulsive," and "no force" should not be inferred solely from coincident-versus-infinite-separation energies without a two-center solution or an asymptotic interaction calculation.

4. **Generic order-three energetics and the numerical evidence are incomplete.**

   Equation (70) gives a genuine two-profile functional for general η, but the paper analyzes binding only on the special slice $\eta = 1$, where Appendix C forces $b = 1$ and equations (72)–(75) reduce the problem to the order-two Abelian system. Figures 2 and 3 show $M = 1$ profiles at $\eta = 0.5$ and $\eta = 2$; they contain no multi-vortex tension comparison. Consequently, the claims in the abstract and at the beginning of Section IV that both families exhibit type-I, type-II, and BPS-like behavior as functions of the mass ratios have not been demonstrated for the generic order-three family.

The authors should compute $E_3(M;\eta,\kappa)$ for at least $M = 1,2$ on a representative two-dimensional parameter grid, compare the relevant same-orientation fusion channel with separated strings, and show where the reduced binding energy changes sign. Because the topology is periodic and the interaction may depend on flux orientation, the physical channel being compared must be stated. If the authors intend to make only the exact $\eta = 1$ observation, the generic claims should be narrowed accordingly.

The numerical work also needs to be reproducible. No solver, radial cutoff, treatment of the regular singular point at $R = 0$, mesh or tolerance, residual, convergence test, or error estimate is given for Figures 1–4. A numerical-method paragraph, tabulated tension values, and machine-readable data would be appropriate. Figure 4 should state which solutions were obtained as local extrema and whether alternative initial guesses were tested.

Finally, equations (40), (41), and (54)–(56) do not give the correct leading cylindrical asymptotics. Linearization yields, up to normalization,

$$1 - K = C_G R K_1(2R), \qquad 1 - h = C_+ K_0(2\kappa R)$$

for the order-two family. For the order-three scalar eigenmodes one similarly finds

$$1 - h_+ \propto K_0(2\kappa R), \qquad h_- \propto K_0(2\kappa R/\eta).$$

Thus the physical exponents are indeed m_G, m_+, m_-, but the prefactors behave as $\sqrt{r}$ for $1 - K$ and $r^{-1/2}$ for the scalar modes. These powers matter when long-range forces are discussed and should not be hidden in constants.

5. **Equation (93) is a holonomy map, not yet an operator-level renormalization-group matching.**

Section VI correctly observes that $p : \widetilde{A}_4 \to A_4$ maps a smooth winding sector to a projected finite-group holonomy. It then concludes that a finite-tension vortex in $[\widetilde{g}]$ is "therefore" described in the infrared by the topological class-sum operator $T_{[p(\widetilde{g})]}$. More is required. A classical dynamical string has a nonzero area term, a definite core, and initially a definite based flux. The operator in equation (92) is instead a gauge-averaged, topological surface insertion. The paper should define the corresponding ultraviolet disorder insertion or boundary condition, explain the subtraction or renormalization of its tension, show how conjugacy-class averaging arises, and identify the strict infrared limit in which deformations of the surface become topological.

The many-to-one nature of the projection makes this issue concrete. The nontrivial $D_1 = [-1]$ sector in Table I projects to C_0 and hence to the identity class operator in the truncated A_4 description. The manuscript should explain what information carried by this stable winding is retained or discarded, and how this is compatible with the global form of the ultraviolet $SO(3)$ theory. In particular, the kernel of $\widetilde{A}_4 \to A_4$ originates in $\pi_1(SO(3)) = \mathbb{Z}_2$ and may be visible to global magnetic or 't Hooft data even though ordinary local A_4 holonomy does not see it. The global $SO(3)$ theory, including any discrete-theta choice and the permitted line-operator spectrum, should be specified before declaring the lift information absolutely lost.

The discussion of Wilson probes in Section V also needs correction. Gauge-invariant Wilson loops in integer-spin $SO(3)$ representations cannot distinguish r_3 from r_3^2: rotations by $+2\pi/3$ and $-2\pi/3$ are conjugate in $SO(3)$ and have identical integer-spin characters. Thus ordinary ultraviolet $SO(3)$ Wilson loops see less than the full distinction between C_2 and C_3. Those classes can be distinguished by the one-dimensional $1'$ and $1''$ representations of the infrared A_4 theory, but the paper must explain how such probes arise from the ultraviolet model, for example through Higgs-dressed lines. The present statement that genuine $SO(3)$ Wilson lines detect the projected A_4 conjugacy class is too strong.

The symmetry is, moreover, an emergent deep-infrared structure of the theory after massive charged fields have been removed, not an exact topological symmetry of the full gauge-Higgs model. The massive vector fields transform as an A_4 triplet and the spin-3 Higgs restricts to $1 \oplus 3 \oplus 3$; their tensor products generate the nontrivial A_4 charge sectors. At finite mass these dynamical fields can spoil topologicity of linked Gukov–Witten surfaces. The manuscript should state the scale-dependent and approximate nature of this matching precisely.

6. **The claimed fusion and Aharonov–Bohm agreement is not demonstrated.**

   Table II is a correct review of the A_4 class algebra for the unnormalized sums in equation (92). It is not a computation of fusion of the smooth $\widetilde{A}_4$ vortices. The ultraviolet flux sectors have a different, finer conjugacy-class structure, definite relative orientations matter before class averaging, and a finite-tension soliton collision is not automatically the operator product of topological surfaces. No ultraviolet class algebra, projected product, or dynamical fusion calculation is presented. The text should therefore not say that the vortices "exhibit the same fusion rule" unless this map is explicitly derived, including its multiplicities and its behavior under the identifications in equation (81).

   Similarly, the Introduction promises a non-Abelian Aharonov–Bohm effect expressed by an "indicator of A_4," but neither the indicator nor a linking observable appears later. For a class-sum operator and an irreducible A_4 Wilson line R, the natural calculation gives an action proportional to

   $$B_R(C) = |C| \, \frac{\chi_R(C)}{\dim R}$$

   with the present unnormalized convention. The authors should display the A_4 character table, evaluate this quantity for $R = 1, 1', 1'', 3$, and match it to the holonomies of the constructed vortex families, taking account of the sign correction in comment 1 and the ultraviolet line-spectrum issue in comment 5. This would turn the proposed relation to non-invertible symmetry into a testable result rather than a kinematical analogy.

   The novelty claims should then be adjusted to reflect the division between new and known material. The classic literature cited in the Introduction already develops non-Abelian discrete flux, Aharonov–Bohm effects, fusion, braiding, and Cheshire charge; A_4 vortex fusion and knots are known in spinor-condensate systems; $SO(3) \to A_4$ Higgs potentials and finite-group Gukov–Witten operators are also established. The paper's potentially publishable advance is the explicit smooth realization and its reduced classical energetics. The assertion that this is the first finite-tension realization should be supported by a direct comparison with the cited local discrete-gauge strings, or qualified.

Minor comments

1. The Introduction says that the vortex solutions are classified by conjugacy classes of the unbroken A_4. Section V correctly shows that the ultraviolet topological sectors are conjugacy classes of $\widetilde{A}_4$; the introductory statement should distinguish this from the projected infrared flux label.

2. The sentence attributing the absence of a mass-charge relation to $\pi_1(G/H) \not\supset \mathbb{Z}$ should be replaced by the more precise statement that the model has no additive integer topological charge that furnishes a global Bogomol'nyi central charge.

3. The paper takes N, M to be positive. It would be useful to include negative labels or explicitly state how anti-vortices, inverse holonomies, and their conjugacy classes are represented. This is especially relevant to the C_2/C_3 orientation convention.

4. Equation (76) should be written as a canonical isomorphism rather than a literal equality between a homotopy group and a subgroup of $SU(2)$. The assumptions on allowed gauge transformations at infinity that lead from based elements to conjugacy classes should also be stated.

5. The Bogomol'nyi completion behind equation (68) and its first-order radial equations should be displayed. This would make both the normalization and the restricted nature of the bound transparent.

6. Statements about inevitable cosmic-string networks, reconnection, and rung formation in Section VII should be presented as motivations for future dynamical study, not consequences already established for this model.

7. The phrase "dynamical matter fields can screen the holonomy" near equation (92) is imprecise. The authors should specify which A_4 representations are dynamical and explain the resulting failure of topologicity or breaking of the emergent surface symmetry.

8. The repeated affiliation spelling "Reserach and Education Center" should be corrected to "Research and Education Center."

9. The paper would benefit from a compact table separating four notions that are currently interwoven: the ansatz label N or M, the $\widetilde{A}_4$ topological class, the projected A_4 flux class, and the infrared class-sum operator. Such a table should incorporate the corrected order-three orientation.

Final assessment

The manuscript contains a promising and concrete result: a simple renormalizable gauge-Higgs model admits explicit smooth embedded representatives associated with the binary tetrahedral topology, and their reduced equations can be studied in detail. The radial field equations, tension normalizations, special $\eta = 1$ reduction, and A_4 class algebra provide a solid technical core. The present version nevertheless conflates embedded stationary solutions with stable topological vortices and a projected holonomy with a full defect renormalization-group matching. Correcting equation (49), analyzing or sharply delimiting stability and binding, and completing the Aharonov–Bohm/fusion matching are necessary before the main claims can be accepted. I would be willing to reassess a substantially revised manuscript addressing these points.