Referee Report

Timelike Entanglement First Law and Linearized Field Equations in Higher Curvature Gravity

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Summary

The manuscript studies a proposed timelike analogue of the holographic entanglement first law in CFTs dual to Lovelock gravity. The construction starts from the Jacobson–Myers (JM) entropy functional and applies the double Wick rotation used in recent work on timelike entanglement entropy. For a hyperbolic timelike region, the authors compare the first variation of the continued JM functional with the expectation value of an analytically continued modular Hamiltonian.

Section 2 reviews the Lovelock action, its AdS vacua, Wald entropy, the JM functional, and the timelike modular Hamiltonian. The relevant AdS branch is described by a root $f_\infty = L^2/\widetilde{L}^2$ of the Lovelock vacuum polynomial. The timelike prescription is stated explicitly as a working analytic continuation rather than a Lorentzian replica derivation.

Section 3 treats a low-energy planar black-brane perturbation in cubic Lovelock gravity. The authors evaluate the variation of the bulk JM functional and its surface completion on the unperturbed hyperbolic hemisphere. They obtain, in equation (3.31),

$$\Delta S = \left(1 - 2\lambda f_\infty - 3\mu f_\infty^2\right) \Delta S_{\text{Einstein}}.$$

The holographic stress tensor is claimed to contain precisely the same factor, leading through equations (3.38)–(3.41) to $\Delta S = \Delta\langle H\rangle$.

Section 4 derives the familiar factorization of the Lovelock equations linearized about a maximally symmetric AdS background:

$$\delta E_{AB}^{\text{Lovelock}} = \mathcal{C}_{\text{L}}\, \delta G_{AB}, \qquad \mathcal{C}_{\text{L}} = 1 - \sum_{m\geq 2} m\lambda_m f_\infty^{m-1}.$$

The same coefficient determines the inverse effective Newton constant in equation (4.10).

Section 5 attempts the arbitrary-order result. It derives a general holographic stress tensor, constructs a boundary completion of the JM functional, argues that the vacuum hemisphere remains extremal for every Lovelock order, and claims that all additional cutoff-surface variations vanish as ϵ^2. Equations (5.31) and (5.77) then give

$$\Delta\langle H\rangle = \mathcal{C}_{\text{L}} \Delta\langle H\rangle_{\text{Einstein}}, \qquad \Delta S = \mathcal{C}_{\text{L}} \Delta S_{\text{Einstein}}.$$

Together with the corresponding Einstein-gravity statement, this is used to claim an equivalence between the family of timelike first laws and the linearized Lovelock equations.

The structural result is plausible. On a nondegenerate Einstein-connected AdS branch, the derivative of the Lovelock vacuum polynomial controls the graviton kinetic term, the stress-tensor two-point normalization, and the linear variation of a constant-curvature JM functional. It is therefore natural that the same coefficient appears on both sides of the proposed first law. The cubic specialization also has the expected form.

In its present version, however, the manuscript does not establish this result at the standard required for publication. There is a pointwise error in the explicit curvature calculation of Section 3, two direct dimensional or cutoff-scaling contradictions in the stress-tensor formulas, an incomplete extremality argument, and incorrect formulas in the arbitrary-order boundary-term calculation. More importantly, the logical step from an on-shell factorization check to an equivalence with the local field equations is not demonstrated without assumptions that appear already to impose the normalizable graviton equations.

Evaluation of novelty and significance

The result is focused and potentially useful, especially if the authors can give a clean arbitrary-order treatment of the JM boundary completion for hyperbolic timelike regions. That is the part most likely to be reusable in other higher-curvature calculations. The physical message that the Lovelock kinetic factor $\mathcal{C}_{\mathrm{L}}$ also governs the timelike first-law variation is interesting to the holography community.

The novelty is nevertheless incremental. The JM functional and ordinary Lovelock HEE are established results; the relation between ordinary entanglement first laws and higher-curvature equations has already been studied; universal cubic-Lovelock corrections to timelike entropy have recently been obtained; and the Einstein-gravity timelike first law on which the present argument relies is due to earlier work by the same collaboration. The new content appears to be the hyperbolic calculation with a non-diagonal excited induced metric, its extension to arbitrary Lovelock order, and the claimed control of the generalized entropy surface term. These contributions could justify publication, but only if the arbitrary-order proof is made technically sound and the conditional status of the timelike prescription is stated precisely.

Recommendation

Major revision. I believe that a correct and appropriately qualified version could be suitable for JHEP or Physical Review D. The universal Lovelock factor is physically well motivated, and several limiting checks support the intended conclusion. Publication should nevertheless await a corrected cubic calculation, a consistent holographic renormalization derivation, a complete treatment of the extremal surface and cutoff boundary terms, and a non-circular statement of what is proved about the linearized equations. The present displayed formulas do not support the main theorem as written.

Major comments

1. **The manuscript checks a common factor, but does not yet prove the advertised equivalence with the local field equations.**

   The standard statement that entanglement first laws imply linearized bulk equations is an off-shell statement. One obtains an identity between the first-law mismatch and a bulk integral of the linearized equations, imposes the first law for a sufficiently rich family of translated, boosted, and rescaled regions, and then infers the local equations. The argument in Section 5 instead starts by imposing bulk transverse-traceless and Fefferman–Graham gauges and then uses

$$\delta\widetilde{h}_{\mu\nu} = z^d \delta\widetilde{h}^{(d)}_{\mu\nu} + \cdots \tag{5.3}$$

with all lower coefficients and the logarithmic term absent. For a generic off-shell metric perturbation, FG gauge alone does not imply this normalizable massless-graviton expansion, nor the transverse and traceless constraints on its coefficient. These are normally consequences of the asymptotic equations and Ward identities. The calculation therefore appears to verify a first-law matching on normalizable graviton data rather than to derive the graviton equations from the first law.

Equations (5.31) and (5.77) establish, if their derivations are corrected, that both sides are $\mathcal{C}_{\mathrm{L}}$ times their Einstein counterparts. This is a useful factorization, but the manuscript must explain which off-shell class is being constrained and quote or reproduce the precise hypotheses of the Einstein result being imported. Ideally the authors should derive an identity of the schematic form

$$\Delta S - \Delta\langle H\rangle = \int_\Sigma \xi^A \, \delta E^{\mathrm{Lovelock}}_{AB} \, \varepsilon^B$$

for arbitrary allowed perturbations, including the boundary terms, and then state the required family of timelike regions. If this is not done, the claim should be weakened throughout to an on-shell consistency check of the timelike first law.

There is also an essential nondegeneracy assumption. At a critical Lovelock point $\mathcal{C}_{\mathrm{L}} = 0$, equation (4.9) loses its kinetic equation while equations (5.31) and (5.77) make the first law hold trivially at this order. The first law then cannot imply $\delta G_{AB} = 0$. The authors should restrict to an Einstein-connected branch with $\mathcal{C}_{\mathrm{L}} \neq 0$, and for a unitary holographic CFT normally $\mathcal{C}_{\mathrm{L}} > 0$, in every theorem and conclusion.

2. **The analytic continuation of the region and of relative entropy is not specified consistently.**

   Section 3 defines the region by

$$-t^2 + |\mathbf{x}|^2 < -T_0^2, \qquad x_{d-1} = 0,$$

   and then states only $t \to -i\tau$ and $x_{d-1} \to i x_{d-1}$. Taken literally, this sends the displayed inequality to $\tau^2 + |\mathbf{x}|^2 < -T_0^2$, which is empty, rather than to the real ball $\tau^2 + |\mathbf{x}|^2 \leq T_0^2$ used in equations (2.14) and (3.40). A continuation of T_0, a different initial inequality, or a nonliteral contour prescription is missing. The anchoring contour, orientation, branch of the complex surface, and all factors of i should be stated explicitly.

   Relatedly, the derivation around equations (2.10)–(2.11) invokes positivity of relative entropy for density matrices on a spatially factorized Hilbert space. A timelike subregion does not define such a tensor factor, and a transition matrix or pseudo-density matrix need not be positive or Hermitian. Thus the ordinary first law cannot simply be carried over by citing positivity. The authors may consistently define an analytically continued identity in the double-Wick-rotated theory, as they partly acknowledge in Section 2.2, but they must distinguish that conditional identity from a first law of genuine timelike density matrices. They should also specify whether ΔS denotes the complete complex timelike entropy or only its real state-dependent part, and demonstrate that the continued JM functional uses the same branch as the complex-extremal-surface prescription at the order being studied.

3. **The explicit cubic calculation contains a wrong local curvature and a dimensionally inconsistent counterterm scale.**

   A direct evaluation of the linearized scalar curvature of the induced metric in equations (3.8) and (3.13) gives

$$\delta\mathcal{R} = \frac{m_z T_0^d \sin^d w}{\widetilde{L}^2} \left[d\cos^2 w \sin^2\theta - d + 2\right].$$

   This is not equation (3.16). The difference between the expression printed in equation (3.16) and the expression above is

$$-(d+1)\left[(d-1)\cos^2\theta - 1\right]\cos^2 w$$

   inside the square brackets. This is an angular harmonic whose integral against $\sin^{d-3}\theta$ vanishes. Consequently, the final integrated answer (3.31) may survive, but the pointwise derivation and every local formula that uses (3.16) are incorrect. The authors should correct the curvature calculation and repeat the bulk and surface variations rather than relying on the accidental angular cancellation.

   There is a second direct problem in equation (3.37):

$$\frac{1}{L''} = \frac{1}{L'} - \frac{3\mu L^4}{5\widetilde{L}^3}.$$

   The last term has dimensions of length, whereas the first two quantities have dimensions of inverse length. Presumably the intended denominator is $\widetilde{L}^5$, but the entire counterterm and stress-tensor calculation leading to equations (3.38)–(3.39) should be rederived after the correction. The authors should also show that the near-boundary blackening function in equation (3.5), with the stated normalization of m_z, solves the cubic Lovelock equations on the selected branch and list the complete FG coefficient needed to verify stress-tensor conservation and tracelessness.

4. **The arbitrary-order holographic stress tensor has a central cutoff-scaling contradiction.**

   Equation (5.29), as printed, states

$$\Delta\widetilde{T}_{\mu\nu} \propto \frac{\widetilde{L}}{z^{d-2}}\delta\widetilde{h}^{(d)}_{\mu\nu}.$$

   Equation (5.30) then multiplies this by $(\widetilde{L}/z)^{d-2}$ and nevertheless obtains a finite boundary stress tensor. The two displayed equations are incompatible. Substituting $\delta\widetilde{h}_{\mu\nu} = z^d\delta\widetilde{h}^{(d)}_{\mu\nu}$ into equation (A.6) shows that the cutoff response must scale as z^{d-2}, not $z^{-(d-2)}$. Even if the inverted power in (5.29) is typographical, it occurs in the central derivation of equation (5.31) and must be repaired explicitly.

   There is a broader gap in the same subsection. The text preceding equation (5.8) says that conformal flatness makes the intrinsic boundary Riemann tensor identically zero. This is false: conformal flatness means that the Weyl tensor vanishes, not the full Riemann tensor. A fixed *flat* boundary source does have zero intrinsic curvature, but an x-dependent normalizable perturbation makes a finite-cutoff slice intrinsically curved. The standard Myers term and holographic counterterms contain intrinsic-curvature contributions. The authors must retain these terms or show by explicit near-boundary power counting that they are subleading and cannot modify the finite state-dependent stress tensor. Similarly, equations (5.18)–(5.19) retain only a volume counterterm; the absence of curvature counterterms and even-dimensional anomaly terms from the variation needs justification. Checks at $m = 1, 2, 3$, followed by explicit conservation and tracelessness checks, would make the general formula convincing.

5. **The proof that the hemisphere extremizes the complete arbitrary-order functional is incomplete.**

   Equations (5.47)–(5.50) package the bulk integrand into a coefficient $C_{m,l}$ multiplying a derivative of $(\gamma^{uu})^l/\sqrt{\gamma_{uu}}$. The coefficient defined in equation (5.48) depends on u through $\dot{F}$, the angular curvature, and the angular measure. The use of equations (5.51)–(5.52) must therefore retain the derivative of $C_{m,l}$, and the contribution at the smooth cap must be shown to vanish. As written, equation (5.52) simply sets the upper endpoint to zero even though $(\gamma^{uu})^l/\sqrt{\gamma_{uu}}$ itself is nonzero there. The full measure may make the complete cap term vanish in the dynamical range, but that is a statement to prove, not assume.

   Even after the boundary cancellation, equation (5.53) must be established for the *entire* bulk and surface functional, including the purely angular Euler-density terms. Equation (5.54) then supplies a conserved conjugate momentum; it does not by itself show that $f = \text{const}$ is the unique regular solution with the stated anchoring data. The authors should evaluate the conjugate momentum at $d \log f/du = 0$, state both endpoint conditions, and analyze degeneracies on special Lovelock branches.

   A shorter and more transparent proof may be available. Equation (A.15) claims that the vacuum hemisphere is totally geodesic in both normal directions. The authors could derive the covariant JM shape equation and show directly that it vanishes on this surface, including the boundary completion. They should also explain why restricting to rotationally symmetric embeddings suffices, or verify the full normal variation rather than only the reduced ansatz.

6. **The cutoff-surface calculation in Section 5.3 contains incorrect curvature formulas.**

   For the metric in equation (5.56), the cutoff sphere has

   $$\sigma_{ab} = \frac{\widetilde{L}^2 \xi^2}{T_0^2 - \xi^2}\, \widetilde{\sigma}_{ab},$$

   with $\widetilde{\sigma}_{ab}$ the unit-sphere metric. Its mixed Riemann tensor is therefore

   $$\mathcal{R}^{\partial\, ab}{}_{cd} = \frac{T_0^2 - \xi^2}{\widetilde{L}^2 \xi^2} \left(\delta^a_c \delta^b_d - \delta^a_d \delta^b_c \right).$$

   Equation (5.67) introduces an additional factor $1/T_0^2$, which is both inconsistent with equation (5.56) and dimensionally wrong. This factor propagates into equations (5.69)–(5.72).

   Equation (5.73) is also not a general linearized scalar-curvature formula:

   $$\delta \mathcal{R}^Y = \frac{4(d-3)}{T_0^2} \delta \widetilde{\sigma}.$$

   For an arbitrary angular metric perturbation, the Palatini variation contains $\nabla_a \nabla_b \delta\sigma^{ab} - \nabla^2 \delta\sigma$ as well as the Ricci contraction, with different sign and normalization. The derivative terms may integrate to zero on a closed sphere, and smooth normalizable data may still give the claimed $O(\epsilon^2)$ suppression, but neither fact justifies the pointwise equation (5.73).

The authors should replace equations (5.67)–(5.76) by a covariant variation, retain radial derivatives that lower the apparent power of z, and show the cutoff scaling after integration. The result should be checked explicitly against the Gauss–Bonnet and cubic boundary terms. Since the vanishing of this contribution is one of the genuinely new claims of the paper, power counting alone is insufficient while the exact formulas used for that power counting are inconsistent.

7. **The dimensional ranges and topological Lovelock endpoint must be treated separately.**

   The cubic formulas contain $(d-5)^{-1}$, so a dynamical cubic interaction requires $d \geq 6$, or bulk dimension $D = d+1 \geq 7$. This restriction is not stated. At lower dimension the cubic density either vanishes or is topological, and the displayed normalization is singular.

   More generally, many equations sum through $\lfloor (d+1)/2 \rfloor$ while also containing $(d-2m)!$. For odd d, the upper term is the Euler density in the even-dimensional bulk and produces the formal expression $(-1)!$. This term does not enter the local equations or the graviton kinetic factor, while its JM contribution is topological only after the appropriate boundary completion is included. The dynamical sums in equations (4.8)–(4.10), (5.27)–(5.31), and (5.77) should stop at $\lfloor d/2 \rfloor$; the topological endpoint should be discussed separately rather than hidden behind an unstated $1/\Gamma(0) = 0$ convention.

   The notation should also be reconciled. Equation (2.2) uses $\mathcal{L}_m$ for the density with m powers of curvature, whereas Section 3 calls the Gauss–Bonnet and cubic densities $\mathcal{L}_4$ and $\mathcal{L}_6$. Without an explicit change of convention, $\mathcal{L}_4$ in the notation of equation (2.2) would be quartic. A single notation and a table of the allowed dimensions would prevent ambiguity.

8. **The paper should identify its new result more sharply relative to the cited literature.**

   The factor $\mathcal{C}_{\mathrm{L}}$ is the standard derivative of the Lovelock vacuum polynomial and is already known to control the linearized graviton and CFT stress-tensor normalization. Constant-curvature JM variations are therefore expected to inherit it. The manuscript should compare its result explicitly with the ordinary Lovelock entanglement-first-law analysis, the recent cubic HTEE calculation, and the cited Einstein timelike first-law paper. In particular, it should say whether equations (5.31) and (5.77) contain new physics beyond analytic continuation, or whether the novel theorem is the vanishing of the generalized boundary completion for normalizable hyperbolic data. The latter would be a useful result if proved carefully. At present the introduction alternates between a broad equivalence claim and a deliberately restricted consistency statement, which obscures the actual advance.

Minor comments

1. The citation key `Sun:2008uf` used in the introduction is absent from the bibliography and renders as an unresolved citation.

2. Equation (5.17) contains an undefined p in the rank of the generalized Kronecker delta. Stray p-dependent subscripts also occur in the curvature decomposition around equations (5.42)–(5.48). All arbitrary-rank contractions should be checked for consistent m-dependent index ranges.

3. In equation (5.57), the statement $\delta\gamma_{ij} = \delta\gamma_{ij}^{\mathrm{pb}}$ appears to omit the $\widetilde{L}^2/z^2$ conformal factor.

The tilded metric perturbation, the full induced perturbation, and their pullbacks should be distinguished consistently.

4. Below equation (4.9), the linearized Einstein tensor should be written entirely with background metrics and background covariant derivatives. Using g_{AB} rather than $\bar{g}_{AB}$ in a first-order definition is needlessly ambiguous.

5. The text before equation (5.8) should say *flat boundary source*, not *conformally flat boundary*. These conditions are not equivalent in dimension greater than two.

6. The reference in the discussion following equation (5.62) is formatted as *equation (A.2)*, although the intended reference appears to be subsection A.2. Equation (A.2) is the generic ADM metric.

7. The two normal directions in Appendix A have different signatures, and the definitions immediately before equations (A.14)–(A.15) use different signs. The normal-bundle convention and the corresponding Gauss–Codazzi signs should be stated once and used consistently.

8. The coordinate change used around equation (A.11) should be written explicitly, including $z(r, \theta)$ and $\xi(r, \theta)$. This would make the total-geodesicity check in equation (A.15) reproducible.

9. Define Ω_n explicitly as the area of the unit n-sphere. The phrase *volume of the d-dimensional unit sphere* after equation (3.31) is ambiguous.

10. The comparison after equations (3.38)–(3.39) should cite equation (3.38) for the coupling-dependent pressure; equation (3.39) gives only the relation between two stress-tensor components.

11. A thorough language and notation edit is needed. Examples include *an an auxiliary field*, *factor, ,while*, *conformal flat*, *there are several extensions would be worth pursuing*, and changing notations for the same induced and boundary metrics.