Referee Report

Holography from number theory: Emergent holographic AdS_2 space-time from exact BPS black hole microstate counting

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Summary

The manuscript proposes a bottom-up route from exact arithmetic formulae for four-dimensional BPS indices to a conformal quantum mechanics and then to an emergent AdS_2 geometry. Section 2 starts from the rational Calogero model in a harmonic trap, separates a collective radial coordinate at fixed angular eigenvalue, and argues that a large-N scaling leaves a DFF_ω degree of freedom localized at the minimum of its potential. Section 3 reviews the $SL(2,\mathbb{R})$ representation of this singular oscillator and the semiclassical localization of its R-vacuum.

Section 4 analytically continues one endpoint of the DFF_ω heat kernel from x_0 to ix_0, reverse Wick rotates the time variable, and divides the result by a product of unnormalized vacuum wave functions. After a further phase and dimensional normalization, equation (4.14) has the form

$$\beta^{-a-1} z^{-a} I_a(z),$$

which is matched to the Bessel factor in a Rademacher summand by setting $\beta = \gamma$. The two applications are the coefficients of η^{-24} for heterotic 1/2-BPS states and of the weak Jacobi form $\varphi_{-2,1}$ for type-IIB 1/8-BPS states. Section 5 then represents the classical Kloosterman phases through prescribed Wilson-line holonomies. For the Jacobi multiplier, it tensors the DFF system with an auxiliary (Q, P) quantum mechanics and engineers a finite Gauss sum from a free-particle propagator, endpoint quadratic kicks, and the c-number term of Appendix C.

Section 6 evolves the R-vacuum in the R-eigenstate/Krylov basis. The exact amplitudes give a negative-binomial distribution, whose double-scaling limit is a Gamma density. Expanding the lattice recurrence yields equation (6.26), with the $SL(2,\mathbb{R})$ covariance in equation (6.27). The authors identify this transformation with the near-boundary isometries of a unit-radius Poincare AdS_2 metric, interpret the continuum Krylov coordinate as a holographic scale, and then identify this metric with the black-hole attractor geometry. Finally, a Schwarzian shift is used to assign $c/12 = z\gamma$ in equation (6.34), after which the standard chiral-CFT interval formula in Appendix A returns z. Appendix B also observes that the Fubini–Study metric on the $SL(2,\mathbb{R})$ coherent-state disk is hyperbolic.

There are worthwhile mathematical observations here. In particular, the singular-oscillator kernel provides a compact representation of the Bessel factor, the multiplier can be written as a finite combination of elementary propagators, and the negative-binomial-to-Gamma limit is transparent. These could support an interesting paper about analytic kernel representations of arithmetic black-hole formulae. The present manuscript, however, repeatedly promotes a representation constructed from the known Rademacher data into an independent microscopic or holographic derivation. Several central steps are also algebraically incorrect or dimensionally inconsistent. Most importantly, the normalization used to obtain equation (6.34) is not the normalization of the dimensionless R generator and inserts the target entropy by hand. Consequently, the advertised emergence of the specific black-hole attractor and the claimed entanglement derivation of its entropy do not presently follow.

Evaluation of correctness and logical structure

Some intermediate formulae are convincing in their proper domain. The DFF spectrum and leading semiclassical localization in Section 3 have the standard form. Subject to a branch choice, the magnitude of the kernel manipulations through equation (4.13) is an algebraic identity. The negative-binomial probabilities in equations (6.9)–(6.11), the Gamma density in equations (6.14)–(6.18), and the finite-ϵ $SL(2,\mathbb{R})$ covariance of the differential equation are also useful. Appendix A ends with the familiar finite-temperature interval formula for a chiral CFT.

The logical links between these ingredients are not established. The fixed-angular-momentum radial reduction does not retain or reproduce the Calogero many-body degeneracy. The object called an amputated Segal–Bargmann heat kernel is not defined as an observable in a fixed Hilbert space, and its normalization is chosen so that the desired Rademacher power appears. The Kloosterman construction explicitly places the modular data into a different auxiliary Hamiltonian for every modular image. The continuum Krylov equation has no smooth nontrivial limit without additional rephasing and scaling. Finally, sharing $SL(2,\mathbb{R})$ with AdS_2 fixes neither a bulk theory nor a black-hole solution. These are structural gaps rather than missing explanatory sentences.

Evaluation of novelty and significance

The potentially novel element is the synthesis of several representations: the complexified DFF kernel, an auxiliary free-particle expression for the Jacobi multiplier, and a Krylov continuum description. That synthesis may be useful to researchers studying arithmetic structures in protected black hole indices. It should, however, be positioned much more precisely. The authors' 2025 JHEP paper already supplied the DFF heat-kernel match to the power corrections. The Calogero motivation, the kinematical relation between conformal mechanics and AdS_2, hyperbolic coherent-state geometry, and the AdS_2 orbifold interpretation of Rademacher/Kloosterman terms all have substantial antecedents among references already cited in the manuscript.

Accordingly, the assertion of the "first known holographic computation of exact BPS black hole entropy in string theory" is not supported by the present argument and is too broad relative both to that literature and to the approximations used here. A defensible contribution would be a carefully defined analytic representation of the Bessel and multiplier factors, together with a clearly labeled information-geometric observation. A true holographic derivation would additionally have to specify a single boundary theory, show that its protected trace predicts the arithmetic coefficients, derive a bulk action and dictionary, and reproduce the charge-dependent attractor fields and entropy without importing them.

Recommendation

Reject in the present form, with reconsideration appropriate for a substantially reconstructed manuscript. The central claims in the title, abstract, and conclusions are not consequences of the displayed calculations. This is not remedied by a limited revision: the Calogero Hilbert-space map must be rebuilt, the kernel and auxiliary quantum mechanics must be defined independently of the desired answer, and the central charge and entropy must be derived from a specified gravitational or boundary system. Alternatively, the authors could produce a substantially more modest and potentially valuable paper centered on kernel representations, while removing the claims that the attractor geometry and Bekenstein–Hawking entropy have been holographically derived.

Major comments

1. **The Calogero Hamiltonian and radial reduction must be corrected, and the counting Hilbert space has not been mapped to the DFF sector.**

   Equations (2.1) and (2.2) place the complete pair interaction $\sum_{i<j} g/|x_i - x_j|^2$ inside the outer sum over i, while reusing the same index. As printed, the pair potential is counted N times. It should be a separate pair sum unless a different convention is explicitly meant.

   There is also a direct error in equation (2.7). Starting with the radial operator (2.4) and setting $R(r) = r^{-(N-2)/2}\widetilde{R}(r)$ gives the inverse-square term

   $$\frac{1}{r^2}\left[\lambda_k + \frac{\hbar^2}{2m}\frac{(N-2)(N-4)}{4}\right].$$

   The manuscript instead writes $r^{-2}[\lambda_k - (N-2)(N-4)/4]$. This has the opposite sign and adds a dimensionless term to the dimensionful λ_k. Equations (2.4)–(2.9) must be rederived with the radial measure, self-adjoint domain, and all factors of m and $\hbar$ stated. Although this measure term is subleading to the displayed N^4 angular eigenvalue, its error prevents the claimed exact reduction and signals that the operator being scaled has not been defined consistently.

   More fundamentally, equation (2.9) contains one collective radial coordinate. It does not produce the "N independent identical DFF degrees of freedom" claimed immediately afterward. Fixing k also freezes the angular/many-body sector in which the Calogero partition degeneracy resides. No trace identity or isometry of Hilbert spaces relates the degeneracy $p(M)$ to this one radial sector, and no dictionary is given among N, M, s, k, the black-hole charges, g_{eff}, and the later DFF parameters g, a. Suppressing the kinetic term in (2.9) gives a classical localization limit, not a normalizable exact zero-momentum state, and it does not justify subsequently using the full finite-mass quantum heat kernel. This bridge is essential to any microscopic claim.

2. **The complexified and "amputated" kernel needs an independently defined physical meaning, and the claimed exactness is not demonstrated.**

   The original DFF configuration space is the half-line. Equations (4.2)–(4.9) continue an endpoint to ix_0, but the manuscript does not define a holomorphic Hilbert space, Segal–Bargmann measure, integration contour, or operator whose matrix element this is. The branch of $x^{a+1/2}$ must be fixed consistently with that structure. Analytically continuing one endpoint of a known kernel is not, by itself, a complete Segal–Bargmann transform.

   The decisive Rademacher powers arise only after dividing by the product of *unnormalized* endpoint wave functions in equations (4.11)–(4.13), and equation (4.14) then removes an additional phase and dimensionful factor. No LSZ, gluing, protected-trace, or path-integral principle selects this amputation. A different normalization of the endpoint state changes the quantity identified with a degeneracy. The authors should define a trace, index, amplitude, or gluing observable whose measure and normalization are fixed before comparison with the Rademacher formula. In its present form the calculation establishes that the known Bessel function can be represented by a tuned singular-oscillator kernel, not that the CQM predicts the black-hole coefficient.

There is also a concrete continuation error. Applying the stated $\beta_E \mapsto i\beta$ to equation (4.9) gives an extra $1/i$, so the prefactor in equation (4.10) is $e^{-i\pi/4}$, not $e^{+i\pi/4}$, when the Bessel argument is $+z$. This propagates into equations (4.13)–(4.14) and cannot be dismissed when exact multiplier phases are later emphasized.

Finally, equation (4.10) uses $\sinh(\omega\beta) \simeq \omega\beta$, while $x_0^2 \sim 1/\omega$. The result is a small-ω, large-z construction. The Rademacher series runs over unbounded $\gamma = \beta$, so the approximation is not uniform in all summands unless an order of limits and an error bound are supplied. Equation (4.21) also tunes ω from the known charge and sector data. The authors should either obtain (4.15) from the exact sinh expression for one fixed CQM or consistently call the result an asymptotic representation. If "exact" is retained, the actual orders $a = 13$ for η^{-24} and $a = 7/2$ for the theta-decomposed $\varphi_{-2,1}$ case, all polar coefficients, signs, and normalizations should be displayed and at least one Fourier coefficient in each theory should be reproduced numerically.

3. **The Kloosterman phases are inserted as sector-dependent input rather than derived from a fixed quantum mechanics.**

   In Section 5.1 the desired holonomies, the congruence $\alpha\delta = 1 \pmod{\gamma}$, and the fractional charges $q = n/\gamma$, $q_m = -1/\gamma$ are prescribed precisely to recover equation (4.18). Thus even the particle charges change with the orbifold sector. Symplectic duality alone does not derive the coprime modular-image sum, the measure on flat connections, or these charge assignments from either string compactification. A convincing origin would start with fixed quantized charges and a specified gauge system and show how its allowed bundles and orbifolds generate the full Kloosterman sum.

   The geometry around equations (5.1)–(5.2) also needs correction. The second metric has its horizon at $r = 1/\gamma$, not at $r = 1$, and the rescaling of the angular period determines whether the result is a smooth disk or a conical quotient. The claimed extension of the boundary time interval and the identification of the dimensionful quantum-mechanical β with the integer γ require a unit of time and a careful boundary proper-time argument. The Wilson-line signs should then be checked with an explicit Euclidean action convention; at present factors of i are assigned both to the connection and to the phase.

4. **The auxiliary multiplier construction is not a well-defined single Hilbert-space system and is partly tautological.**

   Equations (5.6)–(5.13) use $P = -i\partial_x$, continuous position kets, and the real-line free-particle kernel. Equation (5.17) then declares the same coordinate to lie in the finite set $\mathbb{Z}/(2r\gamma\mathbb{Z})$. A particle on a continuous circle has a periodized kernel with winding images and quantized momenta; a finite cyclic Hilbert space instead does not carry the canonical pair used in (5.6). The manuscript supplies neither construction, its measure, nor normalized versions of the distributional states in equation (5.18). Consequently the matrix element (5.19) is only a formal way of writing the finite Gauss sum.

   There is an internal sign inconsistency as well. Equation (5.11) describes positive endpoint potentials, but $e^{-i\int H dt}$ would then give negative phases, whereas equation (5.12) uses positive phases and equation (5.20) accordingly writes negative endpoint terms. Endpoint delta functions also require a time-ordering or half-weight convention.

Most importantly, α, δ, γ are placed directly in those kicks, and Appendix C defines $\varphi(t)$ by explicitly inserting the Dedekind sum whose Rademacher phase is to be explained. Adding a c-number whose integral is the desired phase can reproduce any multiplier. This is a kernel representation, not a Hamiltonian origin, unless the authors derive a single, arithmetic-data-independent auxiliary system whose spectrum or path integral generates the modular matrices and their sum.

5. **The Krylov double-scaling limit has no smooth nontrivial quantum dynamics in the form used for the spacetime interpretation.**

   The discrete probabilities and their Gamma-density limit are sound, but the amplitudes behave differently. With equation (6.12), fixed T requires $t \sim \epsilon^{-1/2}$, and equation (6.25) has a phase diverging as $\epsilon^{-1/2}$. The wave function therefore has no strong smooth limit. At the same time the Hamiltonian coefficient $C = \epsilon\omega$ in equations (6.26)–(6.28) vanishes, so the strict continuum equation is trivial on smooth wave functions. The rapidly oscillating special solution compensates for this only through a singular WKB gradient; it does not define a general continuum Hilbert-space evolution.

   The initial condition in equation (6.25), $\psi_\epsilon(r, T = 0) = 0$, is also incompatible with $C_0(0) = 1$ and with unit normalization. The limiting initial datum is a distribution supported at $r = 0$. A revision must give a rephased variable with a convergent limit, its measure and self-adjoint operator, the boundary condition at the origin, and a finite time-evolution generator. It should also explain why Section 6 assumes $2r_0 \in \mathbb{N}$, since for the $a = 7/2$ example one has $2r_0 = 9/2$; the Gamma-ratio asymptotic itself does not need integrality.

   Nor has $r = \epsilon n$ been shown to be an energy scale of the Poincare Hamiltonian. It is a scaled label in the R representation. Under equation (6.27) it scales like a radial length, while a boundary energy scales inversely. In the chosen AdS_2 metric $r = 0$ is the ultraviolet boundary, but in the lattice construction it is the ground end $n = 0$. The UV/IR dictionary should be derived rather than asserted, and the relation of this r to the coherent-state disk coordinate in Appendix B should be given.

6. $SL(2,\mathbb{R})$ **covariance does not derive the black-hole AdS_2 attractor.**

   Equations (6.27)–(6.31) show that the continuum differential equation has a Schrodinger-group action compatible with the near-boundary transformations of Poincare AdS_2. DFF conformal mechanics was built to have this group. Selecting a familiar metric with the same kinematical symmetry neither fixes the bulk metric uniquely away from the boundary nor determines its radius, dilaton, Newton constant, electric fields, or boundary conditions. No two-dimensional action is written, no spherical reduction is performed, and no field equation or charge-radius relation is checked. The sentence after equation (6.31) that identifies the metric with the attractor geometry is therefore an assumption, not a result.

   This omission is physically important because the two examples do not share an unspecified universal two-derivative reduction. The heterotic 1/2-BPS system counted by η^{-24} is the small-black-hole case in which higher-derivative string corrections are essential to a regular AdS_2 throat, whereas the type-IIB 1/8-BPS large black hole has a different attractor system. A holographic claim must specify the appropriate effective action and full near-horizon fields for each case and demonstrate a dictionary beyond their common $SL(2,\mathbb{R})$ isometry. Likewise, the standard hyperbolic Fubini–Study metric in Appendix B is information geometry on a coherent-state family; a matching group and curvature sign do not make it the physical black-hole spacetime.

7. **The central-charge relation (6.34) is dimensionally incorrect and installs the desired entropy.**

   The representation defined in equations (3.3)–(3.4) has

   $$R|0\rangle = \hbar r_0 |0\rangle, \qquad 2r_0 = 1 + a.$$

   If R generates a dimensionless Euclidean angle, its dimensionless vacuum value is $\langle R\rangle/\hbar = r_0$. The text before equation (6.34) instead divides by $\hbar\omega$ "on dimensional grounds." Since R has units of action, this quantity has units of time, not the units of a central charge. The extra $1/\omega$ is precisely what changes

   $$\frac{\langle R\rangle}{\hbar} \simeq \frac{\omega\beta z}{2} \quad \text{into} \quad \frac{\langle R\rangle}{\hbar\omega} = \frac{\beta z}{2},$$

   and hence produces $c/12 = \gamma z$. Without that unjustified factor, the Schwarzian comparison would give $c/12 = 2r_0 = 1 + a$ exactly, or $c/12 \simeq \omega\beta z$ in the semiclassical approximation. This is fixed by the modular weight and is independent of the black-hole charge, whereas γz is the charge-dependent leading entropy.

   The same inconsistency appears explicitly in Appendix B. Equation (B.4) states $2r_0 = \omega T z$, with T undefined there. If $T = \beta = \gamma$, equations (3.4) and (4.21) instead give

   $$2r_0 = 1 + a, \qquad \omega\gamma z = \sqrt{a^2 - \tfrac{1}{4}},$$

   which are not equal. Dividing the information metric by ωT then does not repair the mismatch; it is an arbitrary rescaling chosen to make the coefficient z. The central charge must be computed from a specified two-dimensional action and its asymptotic charge algebra, with the normalization and Schwarzian sign stated. The claim that bulk two-dimensional gravity "is by itself a chiral 2D CFT" also requires a much more precise statement of which Hilbert space and stress tensor are meant.

8. **Appendix A does not provide an independent entanglement calculation of the Bekenstein–Hawking entropy.**

   Equation (A.6) is the standard interval formula, but equation (A.8) computes $\partial S_{\text{ent}}/\partial l$, which is an entropy density. It is not the total thermal entropy; the extensive interval length l has been dropped. Setting $\widetilde{\beta} = 2\pi\gamma$ then gives $c/(12\gamma) = z$ only because equation (6.34) has already defined $c = 12\gamma z$. Thus equation (A.10) algebraically restates the desired answer rather than deriving it.

   No physical interval length, cutoff, extremal temperature, Newton constant, replica construction, RT/QES surface, or generalized entropy is identified. Moreover, the bulk radial variable $x = \ln r$ is treated as the spatial line of an ordinary factorizing chiral CFT even though the proposed system is a diffeomorphism-invariant gravitational theory. The manuscript must specify the factorization and bulk/boundary dictionary that make this interval entropy meaningful. It must also explain why a high-temperature cylinder limit computes the entropy of the zero-temperature BPS sector. Absent such a derivation, the conclusion should be limited to the formal observation that, after choosing the central charge to contain $z\gamma$, a standard CFT identity returns z.

9. **The arithmetic objects, limits, and comparison with existing holographic work must be stated more accurately.**

   The Fourier coefficients at issue are protected helicity supertraces or indices, not automatically positive Hilbert-space degeneracies. The manuscript should state the sign convention, primitive-charge and divisor conditions, single-center versus multi-center content, and possible hair factors for each example. This distinction is especially relevant because the proposed kernel is a complex amplitude; equating it with a state count requires a no-cancellation argument.

   The current notation "up to an overall numerical constant" is incompatible with repeated claims of an exact derivation. A revised manuscript should retain the polar data, overall factors, multiplier phases, and charge domains, and demonstrate an end-to-end coefficient rather than only a generic Bessel shape. It should also compare its claim explicitly with the existing AdS_2 quantum-entropy-function/orbifold interpretation represented by the Dabholkar–Gomes–Murthy, Murthy–Reys, and Sen references, as well as with the authors' prior DFF paper. This comparison is needed to isolate what the new kernel representation predicts that was not already present in the Rademacher input.

Minor comments

1. In the Gaussian displayed as equation (4.5), the factor $m/\hbar$ from equation (4.1) has been dropped. The cancellation remains true because $x^2 + y^2 = 0$, but the displayed continuation should be corrected.

2. Equation (6.26) writes $\psi_\epsilon(t, x)$, while the differential operator and all surrounding discussion use r.

3. The cutoff relation in equation (A.4) is missing a factor of two in the exponent: from $z = e^{2\pi\sigma/\widetilde{\beta}}$, one obtains

   $$\left|\frac{dz}{d\sigma}\right| = \frac{2\pi}{\widetilde{\beta}} e^{2\pi u/\widetilde{\beta}}.$$

   The subsequent geometric-mean exponent and the final formula (A.6) have the standard form, so the intermediate line is internally inconsistent.

4. Several symbols are overloaded in ways that obscure the proposed dictionary: r is a Calogero radius, the Jacobi level $r = 3$, and a Krylov coordinate; a is a Bessel order and an $SL(2, \mathbb{R})$ matrix entry; C is both a multiplier normalization and $\epsilon\omega$; T is a continuum scale in Section 6 and is undefined in Appendix B; and α is both a modular datum and a coherent-state coordinate. These should be separated.

5. The DFF Hamiltonian on the half-line should state the self-adjoint extension or boundary condition at $x = 0$, even if the Friedrichs extension is intended throughout.

6. Please correct "obtained form" to "obtained from," "the the amputated," the duplicated "in" in the $\mathcal{N} = 8$ paragraph, "net contribute," "accompained," "upto," "on a energy scale," the extra period after equation (6.34), and "an Euclidean."