# Referee Report

Recursion Relations for Classical Gravity

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## Summary

The manuscript develops a recursive representation of the classical two-body problem in post-Minkowskian (PM) gravity. The authors use the gothic metric density, introduce an auxiliary inverse metric so that inversion need not be expanded at every order, and couple the resulting Einstein equations to two point-particle geodesics. Section 3 then replaces an ordinary Fourier transform by a doubled-coordinate decomposition adapted to the two worldlines. Products of fields become convolution brackets, while the operator $\mathcal{E}$ organizes the evaluation of metric currents on perturbed trajectories. The graviton recursion is summarized in equations (4.1)–(4.6), the matter recursion in equation (4.19), and the impulse in equations (4.24)–(4.26).

Section 5 classifies the two-loop integrations needed at 3PM into Family I, Family II, and radiation ("mushroom") integrals. Some integrals are evaluated as nested one-loop integrals. For Family II the authors use IBP reduction and a threshold expansion in $x = \gamma - 1$, separated into potential and radiation regions. Sections 6 and 7 recover the standard 1PM impulse and Westpfahl's 2PM result. Section 8 applies the construction at 3PM, where retarded propagators are intended to incorporate radiation reaction, and presents final "conservative" and "dissipative" impulses in equations (8.51) and (8.52).

This is a potentially useful organization of the classical field equations. The coupled current recursions, the doubled Fourier bookkeeping, and the effort to expose which integrations factorize could be valuable to researchers working on high-order PM dynamics. The successful recovery of familiar 1PM and 2PM results is an encouraging check. At 3PM, however, the paper does not derive a new observable; its publication value rests on the correctness, clarity, and prospective efficiency of the method. In its present form the central 3PM derivation is not sufficiently reliable. I find an apparent advanced/retarded sign reversal, inconsistent mass and momentum powers in the displayed 3PM formulas, an undefined hybrid dimensional scheme, and several contradictions in the master-integral results. These are not merely stylistic issues: they occur precisely in the steps that are supposed to establish the radiation-reacted answer.

## Evaluation of correctness and logical structure

The early construction has a coherent formal outline. The 1PM metric and worldline currents reproduce the expected transverse impulse, and the 2PM answer contains both the transverse deflection and the longitudinal iteration required by the mass-shell condition. It is also reasonable that a retarded initial-value solution of the classical equations contains radiation reaction without a separate conservative/dissipative split at the level of the equations.

The manuscript nevertheless does not yet provide a checkable chain from the formal recursions to equations (8.51)–(8.52). In particular, the causal sign of the on-shell graviton denominator changes after the Family-II variable shift; equations (8.5), (8.11), (8.38), and (8.39) cannot all be true with the definitions given; and the threshold master integrals disagree between Section 5 and Appendix B in their dimensions and signs. The claimed four-dimensional treatment of the field equations is also mixed with $(4-2\epsilon)$-dimensional tensor algebra in a calculation containing $1/\epsilon^2$ poles. Agreement of the final expression with a known result does not by itself resolve these internal inconsistencies.

## Evaluation of novelty and significance

The potentially novel contribution is methodological rather than a new piece of two-body physics. That contribution should be stated much more sharply. The Introduction cites prior work on off-shell gravity recursions, all-order perturbations of general relativity, and, especially, "Binary Black Holes and Quantum Off-Shell Recursion," which also treats binary scattering through 3PM. The present paper should explain in a dedicated comparison what is new: for example, whether it is the purely classical initial-value formulation, the coupled matter-current recursion, the doubled-coordinate decomposition, the implementation of retarded boundary conditions, or the organization of the integrals.

The claim of a novel iterated-one-loop integration method is presently too broad. Section 8 explicitly states that the mixed modes do not admit such a factorization and instead require IBP reduction to known two-loop masters. Because the observable is already known, a convincing demonstration of reduced complexity or improved scalability is important. Term counts, symbolic run times, machine-readable recursions and IBP reductions, or an explicit higher- order example would make the significance far easier to assess.

## Recommendation

**Major revision.** I would not recommend publication of the present version, but the underlying current-recursion framework appears sufficiently interesting to merit reconsideration after a substantial technical revision. At a minimum, the causal sign and dimensional scheme must be fixed, the 3PM mass and momentum bookkeeping must be made internally consistent, the master integrals must be derived and validated in a reproducible way, and the final impulse must pass explicit mass-shell and radiated-momentum balance checks.

## Major comments

1. **The retarded prescription appears to change sign in the Family-II shift, and the all-order worldline recursion does not state its boundary prescription.**

   Let $p = \ell - k_1 - k_2$. Equation (5.3) defines the metric denominator as

   $$p^2 - \mathrm{sgn}(p^0) i0_+.$$

   In the frame used in equations (5.6)–(5.7), the two cut conditions give $k_1^0 = 0$, $k_2^0 = \sqrt{1-\gamma^{-2}}\, k_2^z$, and $\ell^0 = 0$. Hence $p^0 = -\sqrt{1-\gamma^{-2}}\, k_2^z$, so the same retarded denominator is

   $$p^2 + \mathrm{sgn}(k_2^z) i0_+.$$

   The positive rescaling of $k_2^z$ in equation (5.30) and the subsequent spatial shift of $k_1$ do not reverse this sign. Nevertheless, equation (5.34) contains $-\,\mathrm{sgn}(\vec{q}\cdot\vec{k}_2) i0_+$. The same sign is then used in the radiation-region and mushroom denominators. Unless an additional unwritten change of variables is intended, this is the advanced rather than the retarded choice in the on-shell channel and can reverse the absorptive and radiation-reaction terms.

The authors should trace the Fourier convention and every energy variable from equation (5.3) to equations (5.34), (5.46), (5.48), and (5.60), then recompute the affected odd and radiation masters and equation (8.52). A numerical contour check at a fixed $\gamma > 1$ would be useful. In addition, equation (4.19) displays $(L_{12} \cdot v_\alpha)^{-1}$ and $(L_{12} \cdot v_\alpha)^{-2}$ without any $i0$ prescription. A formal recursion for a scattering initial-value problem is incomplete until the incoming boundary condition and possible homogeneous solutions are specified at this level, rather than inserted only in selected later integrals.

2. **The 3PM mass normalization and Fourier scaling are internally inconsistent.**

Equation (8.5) factors $m_1 m_2^3$ outside the Fourier integral, while equation (8.11) defines the integrand appearing there with another factor $m_1 m_2^3$. Read literally, these equations give the square of the desired mass factor. The same problem occurs between the mass factors in equation (8.6) and the definition of the $m_1^3 m_2$ integrand in equation (8.27). Later formulas tacitly use only one factor, so the notation is not a harmless alternative convention.

There is an even sharper contradiction in the $m_1^3 m_2$ sector. Equations (8.32) and (8.37) both contain a nonlocal longitudinal term proportional to $|\ell| w_1^\mu$. Their stated sum, equation (8.38), instead contains $|\ell|^2 w_1^\mu$. This conflicts with the manuscript's own scaling argument in equations (8.8)–(8.9): in the transverse Fourier plane, $|\ell|$ produces the required $|b|^{-3}$ behavior, whereas $|\ell|^2$ is analytic and gives only a contact distribution. Equation (8.39) nevertheless Fourier transforms the printed $|\ell|^2$ term into a nonlocal $|b|^{-3}$ term. Equation (8.39) also omits the entire $m_1^3 m_2$ factor, although equation (8.51) restores it.

The authors should replace the present mixture of mass-stripped and mass-dressed quantities by one convention, audit the mass dimension and $|\ell|$ power of every term, and regenerate equations (8.5)–(8.39) from the symbolic output. The corrected intermediate expressions, not only the known final answer, should be supplied in a machine-readable ancillary file.

3. **The hybrid four-dimensional/dimensionally regulated scheme is not defined and can change finite terms.**

Immediately after equation (2.23), the paper specializes the classical field equations to four dimensions while retaining $D$-dimensional loop integrals, and asserts that the difference from a fully $D$-dimensional treatment cannot survive because there is no room for classical subtractions. This assertion is not valid without proof. The master integrals displayed later contain simple and double $1/\epsilon$ poles, so an evanescent $(D-4)$ numerator or trace can multiply a pole and contribute to the finite long-range result even when no counterterm is introduced.

The actual calculation also seems different from the prescription stated in Section 2.2. Equation (7.8) already contains both $\eta^{\mu\nu}$ and $\widetilde{\eta}^{\mu\nu}$, while Appendix C.1 only later declares that $\widetilde{\eta}^{\mu}{}_{\mu} = 4-2\epsilon$. The contractions between these two metrics, the dimension of the external velocity subspace, and the Ward identities in this scheme are not defined.

The field equations should either be derived consistently in $D = 4 - 2\epsilon$ and reduced to four dimensions after integration, or the authors should specify a precise four-dimensional-helicity-type scheme and derive its conversion to conventional dimensional regularization. They should also show explicitly that the finite nonanalytic parts of the 2PM and 3PM impulses are unchanged. This issue is central to a paper whose main claim is an all-order algorithm.

4. **The Family-II master integrals are not internally consistent or sufficiently established.** There are several direct checks that fail. In $d = 3 - 2\epsilon$ spatial dimensions, the masters $f_4^{\mathrm{II},e} = J^{\mathrm{II}}_{2,1,1,0,0,0,0,1,1}$ and $f_5^{\mathrm{II},e} = J^{\mathrm{II}}_{1,1,2,0,0,0,0,1,1}$ have mass dimension $-2-4\epsilon$. Equation (5.43) prints $|\ell|^{-4\epsilon}$ for both, missing $|\ell|^{-2}$; Appendix B.2 instead gives $|\ell|^{-2-4\epsilon}$. Conversely, $f_3^{\mathrm{II},e} = J^{\mathrm{II}}_{1,1,1,0,0,0,0,1,1}$ has dimension $-4\epsilon$, as in equation (5.43), but its reduction in Appendix B.2 inserts an extra $|\ell|^{-2}$.

   There is also a sign contradiction for $f_8^{\mathrm{II},e}$. Appendix B first gives the relevant $\overline{\mathrm{II}}$ master a positive leading term $+1/(2\epsilon^2)$, and then writes $f_8 = (\gamma^2-1)^{-1}[f_3^{\overline{\mathrm{II}},e} + \cdots]$, but the very next displayed result has a negative $-1/[2\epsilon^2(\gamma^2-1)]$. Equation (5.52) additionally uses $\Gamma(n_1)\Gamma(n_2)$ in a formula whose integral powers are denoted $n_1', n_2'$.

   More generally, Appendix B.2 repeatedly states that the threshold series was truncated at $x^{10}$, after which finite series are promoted to exact expressions involving $\operatorname{arccosh}\gamma$, logarithms, and dilogarithms. No reconstruction ansatz, differential equation, analytic recurrence, error estimate, or numerical validation is provided. Even the claim $f_1^{\mathrm{II},P,e} = 0$ is inferred there from ten orders rather than proved (although a direct scaleless-integral proof may exist). The authors should provide exact IBP or differential equations with boundary data, explain the resummation uniquely, and test the closed forms numerically at several relativistic values of $\gamma$. The reductions and sufficient series data should be released as ancillary material.

   The potential-region series itself also has an unresolved sign convention. After the changes of variables leading to equation (5.40), the two factors from the linear propagators cancel and the displayed expansion is in $+x^m$. Appendix B.2 instead uses $(-x)^m$ in the reductions leading to several masters, including the formulas printed as equations (B.15), (B.24), (B.27), and (B.29), while the reduction leading to equation (B.20) uses $+x^m$. The manuscript should state whether another variable transformation is being made and make this convention uniform before any resummation is trusted.

5. **The distributional identities used to reduce the cut integrals must be corrected and derived precisely.**

   Equation (5.2) defines $\widehat{\delta}^{(n)}$ as the ordinary $n$-th derivative. Equation (5.25) then states

$$\widehat{\delta}^{(n+1)}(k\cdot v) = \frac{(-1)^n}{n!}(-v^\mu \partial_{k^\mu})^n \widehat{\delta}(k\cdot v).$$

   For $n=0$ this says $\widehat{\delta}' = \widehat{\delta}$, so it cannot be correct with the definition in equation (5.2). It appears that a normalized cut-propagator index has been confused with a derivative order. The notation in equation (5.1), which prints $n_{8-1}$ and $n_{9-1}$ rather than $n_8 - 1$ and $n_9 - 1$, compounds the problem. Since equation (5.26) is obtained by integration by parts from these distributions, all associated factorials and signs should be rederived.

   The Family-I factorization also needs a distributionally valid statement. With $X_i^\pm = X_i \pm i0$, equation (5.17) cannot literally satisfy $X_1^\pm + X_2^\pm + X_3^\pm = 0$, because the infinitesimals add. Moreover, the right-hand side of equation (5.18) contains three scalar delta functions even though the momentum-conserving delta function in equation (5.16), together with $\ell\cdot v_2 = 0$, already imposes $X_1 + X_2 + X_3 = 0$. The next line effectively uses only two independent delta functions.

The correct identity should be written as an identity of distributions including the conservation delta and its Jacobian, then checked against a test function. The relation $f_3^{\mathrm{I},+} = -2f_3^{\mathrm{I},-}$, and hence the Family-I contribution to both unequal-mass 3PM sectors, depends on this step.

6. **The radiation interpretation and the final physical checks are incomplete.**

   Section 5.3 states that 3PM is a borderline case "where energy is not radiated away." This is incorrect as a statement about unbound two-body scattering: the leading radiated four-momentum is nonzero at order $G^3$. Indeed, the manuscript's cited worldline-QFT and PM-EFT radiation-reaction papers compute precisely this quantity. If the authors intend a narrower statement about a particular component or intermediate region, it must be spelled out; otherwise the sentence should be corrected.

   The paper should use the known radiated momentum as a stringent independent test. More urgently, the displayed formulas already fail the individual mass-shell test. Write $\Delta p_1 = Gq_1 + G^2q_2 + G^3q_3 + \cdots$, use equations (6.12), (7.19), (8.51), and (8.52), and define

   $$K = \frac{3\pi m_1^2 m_2^2(m_1+m_2)(2\gamma^2-1)(5\gamma^2-1)}{2(\gamma^2-1)|b|^3}.$$

   The transverse 1PM and 2PM terms give $q_1 \cdot q_2 = -K$. From $v_1 \cdot w_1 = 0$ and $v_1 \cdot w_2 = -1$, the longitudinal part of equation (8.51) gives $p_1 \cdot q_3 = -K$; equation (8.52), containing only $b^\mu$ and $w_1^\mu$, contributes zero to this contraction. The order-$G^3$ condition

   $$p_1 \cdot q_3 + q_1 \cdot q_2 = 0$$

   therefore gives $-2K$, rather than zero. By contrast, the displayed 1PM and 2PM formulas satisfy the order-$G^2$ shell condition, so this is not a global impulse-sign convention. Reversing either the transverse sign in equation (7.19) or the relevant longitudinal sign in equation (8.51) would remove this particular mismatch; the authors must locate the erroneous sign from the derivation and recheck the final result.

   After correcting this failure, the authors should derive the impulse of particle 2 by exchange and verify

   $$\Delta p_1^\mu + \Delta p_2^\mu + P_{\mathrm{rad}}^\mu = 0$$

   through 3PM, with all normalization and sign conventions displayed. It should also display the individual mass-shell identities order by order, for example $2p_1 \cdot \Delta p_1^{(2)} + (\Delta p_1^{(1)})^2 = 0$ and $2p_1 \cdot \Delta p_1^{(3)} + 2\Delta p_1^{(1)} \cdot \Delta p_1^{(2)} = 0$. These checks would directly expose mistakes in the longitudinal $w_i^\mu$ terms.

   Finally, Section 8 says that the conservative sector is defined "rather arbitrarily." A physical result should instead define the split by a time-symmetric/time-antisymmetric Green function, or map explicitly to the convention used in the comparison literature. Since the Introduction claims agreement with the 3PM scattering angle while the paper displays an impulse, the conversion to the angle and the treatment of radiation reaction should either be shown or the claim should be narrowed.

7. **The novelty and claimed all-order efficiency require a direct comparison and reproducible evidence.**

   The present discussion puts several closely related recursion approaches into one sentence. A revised manuscript should compare, item by item, with the off-shell current recursion of the cited

2022 JHEP paper, the cited all-order GR perturbation formulas, and the cited 2023 work on binary black holes and quantum off-shell recursion. The comparison should identify the variables, matter treatment, boundary conditions, observable, and loop families in each approach. Retarded propagators are a correct implementation of a classical initial-value problem, but their role in 3PM radiation reaction is already established in the worldline and EFT references; the new point here is their realization within the proposed recursion.

The scope of factorization should also be quantified. Which pure-mode integrals reduce to nested bubbles or triangles? Which mixed-mode integrals still require a conventional two-loop IBP reduction? How many terms and masters occur at 1PM, 2PM, and 3PM, and how does this compare with a worldline calculation? Without such information, the Conclusion's statements that a "precise solution is provided to arbitrarily high order" and that only algebraic equations remain are too strong. What is presently supplied is a formal recursion; the paper itself acknowledges that integration is the main and rapidly growing bottleneck.

8. **The physical scope is that of minimally coupled nonspinning point particles, not generic black holes at arbitrary order.**

   The abstract calls the calculation scattering of two black holes, whereas the Introduction correctly explains that the finite-order construction contains point sources and no horizon. The displayed equations include neither spin nor finite-size operators, tidal response, horizon absorption, or internal dissipation. At the order studied, the universal point-particle sector may indeed reproduce the nonspinning black-hole impulse, but that is a domain-of- validity statement and should replace the stronger ontological claims.

   The assertion that effective-field-theory notions can simply be discarded is also misleading. EFT is not required as a computational language for the minimal 3PM calculation, but matching is what distinguishes a black hole from another compact object and organizes finite-size and absorption effects. Likewise, a successful resummation of the one-body Schwarzschild solution does not establish convergence of the dynamical two-body PM series, nor prove that an all-order sum describes two outgoing Schwarzschild black holes after radiation or a possible close encounter. The abstract, Introduction, and Conclusion should consistently state the point-particle assumptions and limit claims of "all dissipative effects" to radiation reaction within that model.

# Minor comments

1. The labels attached to the matter recursion and to the integrated geodesic equation are duplicated, as are the labels attached to two distinct 2PM source formulas. This produces multiply-defined-reference warnings and can direct a reader to the wrong equation. All equation labels and references should be made unique.

2. Define $\widehat{\delta}$ explicitly (in particular, whether it denotes $2\pi\delta$) and collect the Fourier-measure conventions in one place. The normalization is otherwise difficult to follow across Sections 3–8.

3. Do not use $\epsilon$ both for the dimensional regulator and for the causal infinitesimal. For example, the denominators introduced below equation (8.14) and several Appendix-B formulas use $i\epsilon$ while the same equations are expanded in $\epsilon$. Use $i0_+$ consistently.

4. The tensor $\widetilde{\eta}^{\mu\nu}$ appears in the main 2PM result before it is defined in Appendix C. It should be defined when first used, together with all contraction rules required by the regularization scheme.

5. The statement following equation (5.22) that an Appendix-B master is "identical" to $f_3^{\mathrm{I},+}$ after removing $(\gamma^2-1)^{-1}$ overlooks the displayed factor of $-1/2$ associated with the opposite linear-propagator prescription. The branch should be named explicitly.

6. Equation (A.2) contains $2^{\epsilon}$ in the Fourier transform of $|\ell|^{\delta}$. Specializing the general result in equation (A.1) to the exponent $\delta$ instead gives $2^{\delta}$. This normalization should be corrected and the Fourier transforms used in Section 8 rechecked.

7. The action discussion promises that a gauge-fixing term will be given later, but no such term appears. Either supply the term and its normalization or state clearly that harmonic gauge is imposed on the equations only after variation.

8. Correct the numerous transcription problems before resubmission, including the literal word "to" in equation (5.30), the stray  in equation (8.41), missing vector indices on several $w_i$ terms, $e^{i\ell b}$ in place of $e^{i\ell\cdot b}$, repeated words, the introductory mixed mode $[\ell_2, \ell_2]$ in place of $[\ell_1, \ell_2]$, and encoded nonbreaking-space artifacts. These are particularly damaging in formulas where signs and powers are already at issue.

9. The paper calls the work of Dlapa et al. cited at the end of Section 8 a "comprehensive review," but that reference is a research article. The original 3PM conservative and radiation-reaction results should be cited directly, and a short convention-by-convention comparison should accompany the assertion of agreement.

10. The bibliography contains entries that are not cited in the text. They should either be used in the discussion of EFT and finite-size scope or removed. The author e-mail information requested by the journal style is also absent.