Referee Report

From Fredholm Determinants to AdS/CFT Observables: A Universal Strong-Coupling Framework

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Summary

The manuscript studies the strong-coupling trans-series of two related quantities in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and its string worldsheet description: the cusp anomalous dimension and the dynamically generated mass gap of the two-dimensional O(6) sigma model. Its organizing idea is to use the one-parameter "tilt" $a = \alpha/\pi$. Away from the physical value $a = 1/4$, the two exponentially small scales Λ_-^2 and Λ_+^2 are distinct. This resolves structures that are obscured when the actions become commensurate in the physical theory. In particular, the manuscript argues that the apparent failure at order Λ^6 of the single-scale ratio observed by Dorigoni and Hatsuda is caused by the first admixture of a Λ_+ sector into the pure Λ_- sequence.

Section 2 reviews two descriptions of the tilted cusp. The first uses the generalized BES integral equations and the functions $f_{0,1}$, $V_{0,1}$, and $U_{0,1}^{\pm}$; the second writes the cusp as the ratio of the matrix-Bessel Fredholm determinants $Z_1(\alpha)/Z_0(\alpha)$. Section 3 relates the BES solution to the differential-equation variables $q_\pm$, W_0, and $W_\pm$. The treatment distinguishes the finite-x expansion from the solution at the origin, parameterized by $c_\pm$, and culminates in the matching formula (3.24). This is an important bridge: it expresses the combination entering the mass gap in terms of the same determinant building blocks that control the cusp.

Section 4 introduces the tilted mass gap by equation (4.1), rewrites it in terms of $a_\pm$ and $c_\pm$ in equation (4.3), and derives its double trans-series. The perturbative sector (4.6), its normalization (4.7), and several initial nonperturbative sectors (4.8) are displayed explicitly. At $a = 1/4$, equation (4.11) reproduces the known perturbative O(6) mass gap, and equation (4.12) agrees with the previously known nonperturbative terms. The pure Λ_- ratio (4.9) and the relation (4.10) between the first cusp sector and the square of the perturbative mass-gap sector suggest a considerably more general structure.

Section 5 reorganizes the determinant trans-series in a basis indexed by (δ^+, δ^-), reviews the determinant shift rule (5.6), the Stokes recurrences (5.8)–(5.9), and the determinant Alien algebra (5.13), and transfers this structure first to the tilted cusp and then to the tilted mass gap. The proposed cusp and mass-gap Alien algebras are equations (5.19) and (5.30). They imply the successive-sector ratios (5.25) and (5.31), the sectorwise cusp–mass-gap relation (5.34), and the compact all-orders formula (5.35). Section 5.4 specializes the construction to the physical value $a = 1/4$.

The central idea is elegant and potentially valuable. Separating the two actions is a natural explanation of the previously puzzling third exponential order, and the origin matching together with the tilted mass-gap construction appears to provide a useful new way of organizing the O(6) result. The physical-limit matches are nontrivial evidence for the proposal. In its present form, however, the manuscript does not establish its strongest advertised conclusions. The Alien algebras on which the all-orders statements rest are explicitly conjectured and checked only to finite order; Appendix C assumes those algebras in the purported proofs. There is also a definite normalization

error in the physical moments in equation (5.38), together with operator ordering errors in Appendix A. These issues should be resolved before publication.

Evaluation of correctness and logical structure

The logic through Section 4 is coherent at the formal strong-coupling level. The generalized BES equations and determinant representation are well motivated by the cited literature, the separate treatment of $q_{\pm}(0)$ addresses a genuine nonuniformity of the finite-x ansatz, and equation (3.24) provides the key connection needed for equation (4.3). The recovery of the first terms of the known physical mass gap in (4.11) is an important normalization check. The agreement of (4.12) with earlier resurgent calculations, including the correction of the stated typo in a previous Stokes constant, is further evidence that the construction captures the intended sectors.

The main logical gap occurs when the manuscript passes from these finite-order results to the complete Alien calculus. Equation (5.19) is introduced as a claim, equation (5.30) is explicitly called a conjecture, and the evidence quoted in Sections 5.2–5.3 consists of analytic checks through finite powers of $1/g$ and numerical checks at two values of a. Appendix C then lists the very algebras (5.19) and (5.30), as well as the ordinary Leibniz and chain rules, among its assumptions. Its induction is useful and appears algebraically sound under those assumptions, but it proves a consequence of the conjectured bridge equations rather than the bridge equations themselves. The same observation applies to the passage from (5.34) to (5.35). Calling the latter an exact relation between observables is therefore stronger than the derivation presently supports.

There is also a direct internal inconsistency in the specialization to the physical moments. From equation (2.17), with the standard Dirichlet beta function, one obtains

$$J_{2k+1} = 2^{4k}\,\beta(2k).$$

Equation (5.38) instead prints $2^{2k}\beta(2k)$. At $k = 1$, the printed formula gives $J_3 = 4K$, whereas the manuscript correctly uses $\mathcal{I}_3 = 16K$ below equation (4.11). The even formula in (5.38) also gives the undefined product $0\,\zeta(1)$ at $k = 1$; $J_2 = -6\log 2$ must either be stated separately or obtained through an explicitly stated limiting prescription. Because the moments feed the shifted sectors (5.39) and all subsequent physical formulae, this is not merely a cosmetic typo: the authors must identify which value was used in every displayed and numerical result and recheck the downstream expressions.

Two further inconsistencies occur already in the defining BES/resolvent conventions. With the wave functions (A.2) and the standard inner product implied by their stated orthonormality,

$$\langle\psi_n|\chi|\psi_m\rangle = \sqrt{nm}\int_0^\infty \frac{dt}{t}\,\chi\!\left(\frac{\sqrt{t}}{2g}\right) J_n(\sqrt{t})J_m(\sqrt{t}) = \frac{1}{2}K_{nm}$$

by equation (1.8). Equations (A.3)–(A.4) would therefore produce, for example, $\frac{1}{2}\cos^2\!\alpha\,K_{oo}$, whereas equation (1.6) contains $2\cos^2\!\alpha\,K_{oo}$. The same mismatch occurs in all four blocks. In addition, equations (2.3) and (2.5) imply that equation (2.8) must involve $f_0(0)V_0(0) + f_1(0)V_1(0)$, not the printed minus sign. These conventions set the normalization of the tilted cusp and the later mass-gap bridge and must be reconciled before the downstream coefficients can be considered verified.

Finally, Appendix A currently does not provide a valid operator derivation as written. The projector in equation (A.6) has an undefined bra index m. More importantly, equation (A.11), $\boldsymbol{\gamma} = (1 + \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{X}})^{-1}\boldsymbol{\mathcal{H}}$, does not imply the right-multiplied identity printed in (A.12). The immediate identity is $(1+\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{X}})\boldsymbol{\gamma} = \boldsymbol{\mathcal{H}}$; equivalently, using the push-through identity, one may use $\boldsymbol{\gamma}(1+\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{H}}) = \boldsymbol{\mathcal{H}}$. The manuscript instead writes $\boldsymbol{\gamma}(1+\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{X}}) = \boldsymbol{\mathcal{H}}$, which would require an unproved commutation property. Equation (A.14) follows from the second correct identity, not from the one printed. Since Appendix A supports the fundamental relation (3.1), the operator order should be corrected and the subsequent integral equations rederived with the corrected sequence.

These problems do not, in my view, warrant rejection. The known physical coefficients and the two-scale organization make the main picture plausible. They do, however, prevent the present version from meeting the standard of a complete proof or an exact resurgent solution.

Novelty and significance

The manuscript concerns a technically sophisticated and active interface among AdS/CFT, integrability, Fredholm determinants, and resurgence. A complete, controlled trans-series for the O(6) mass gap, together with a precise relation to the cusp anomalous dimension, would be of clear interest to the JHEP readership. The explanation of the Λ^6 breakdown in terms of colliding Λ_- and Λ_+ sectors is particularly useful, because it converts an apparent failure of the earlier ratio rule into a transparent feature of the physical specialization.

The novelty must nevertheless be delineated more carefully. By the manuscript's own attribution, the determinant shift rule, Stokes recurrences, and determinant Alien algebra in Section 5.1 are results of earlier work. Much of the tilted-cusp discussion in Section 5.2 is likewise attributed to the recent Boldis and Bajnok–Boldis–le Plat papers. The clearest new components here appear to be the origin solution and matching through $c_\pm$ in Sections 3.2–3.3, the definition and determinant representation of the tilted mass gap in (4.1) and (4.3), the generic-a mass-gap sectors, the two-scale explanation of the physical mixing, and the proposed mass-gap algebra and cusp–mass-gap relations in (5.30)–(5.35). An explicit itemized comparison with the recent papers by the same collaboration is needed, especially because the manuscript itself states that one of them already gives a compact notation involving both the cusp and the mass gap.

For generic α, the status of $m(\alpha)$ also affects the significance claim. Equation (4.1) is introduced as a definition, and the manuscript establishes that it reduces to the physical O(6) gap at $a = 1/4$. It does not identify a deformed sigma model, TBA quantity, worldsheet excitation, spectral problem, or other independent observable whose mass is $m(\alpha)$. Nor is a uniqueness principle given for this continuation away from $a = 1/4$. As currently presented, the generic object is best understood as an auxiliary generating deformation. That is still a useful construction, but it should not be called a physical AdS/CFT observable without an additional characterization.

Recommendation

Major revision. I would be willing to reconsider a substantially revised version. The determinant organization, the physical-limit checks, and the explanation of the mixed third exponential order are sufficiently interesting for publication. Before that stage, the authors should either prove the proposed Alien algebras or consistently state the resulting all-orders relations as conjectural formal identities; correct the physical moment formula and audit its consequences; repair the Appendix A operator derivation; and define the resummation, reality, parameter-domain, and resonance prescriptions needed to interpret the physical result.

Major comments

1. **The headline all-orders claims are conditional on the central conjecture.** The abstract states that the paper determines the complete strong-coupling trans-series, constructs an Alien calculus generating all sectors, proves conjectured ratio relations, and derives an exact all-orders cusp–mass-gap relation. The body of the paper is more cautious. Equations (5.19) and (5.30) are proposed on the basis of the determinant analogy and finite-order tests. Appendix C then assumes these equations in order to prove (5.25) and (5.31). Thus the induction does not establish the main new Alien algebra; it establishes the ratio rules if that algebra and the stated differentiation properties hold. Equation (5.34), and consequently (5.35), inherit the same conditional status.

   The authors should derive (5.19) and especially (5.30) directly from the determinant representation and a precisely defined resurgent calculus. For the mass gap this must include the square root in (5.29), its branch, and the action of the Alien derivatives on the g-dependent normalization and on the composite expression. If such a derivation cannot be supplied, the paper should explicitly label (5.19), (5.30), (5.31), (5.34), and (5.35) as conjectural formal trans-series relations supported by the displayed tests. The abstract, Introduction, Section 6, and conclusion should then replace "prove," "complete," and "exact" by language matching that status. It would be acceptable and still interesting to present a well-tested conjectural structure, but the distinction is essential for assessing correctness.

2. **Equation (5.38) has a definite normalization error and an ill-defined endpoint case.** Substitution of $a = 1/4$ into the polygamma expression (2.17) gives $J_{2k+1} = 2^{4k}\beta(2k)$, not the odd-moment formula printed in (5.38). The $k = 1$ case exposes the error immediately: the correct value is $J_3 = 16K$, in agreement with the value used below (4.11), while (5.38) gives $4K$. Moreover, the even formula in (5.38) should be restricted to $k \geq 2$, with $J_2 = -6\log 2$ stated separately, unless a limiting continuation of the product with $\zeta(1)$ is defined.

   Please correct (5.38), define the convention for the Dirichlet beta function, and recheck (5.39), the ratios in the physical subsection, and every numerical specialization that uses J_n. The revised manuscript should say whether the calculations underlying (4.11)–(4.12) and the later ratios used the correct moments and only the printed summary is wrong. At least one explicit substitution, for example the derivation of the K-dependent term from J_3, would make the audit transparent.

3. **The resurgent objects and the meaning of "resumming" equation (5.34) must be defined.** Section 5.1 says that (5.1) corresponds to the upper lateral resummation $\mathcal{S}_+$, but the paper does not specify the Borel variable and its normalization, the relevant Stokes rays, whether $\Delta_j^{\pm}$ are pointed or unpointed Alien derivatives, or the precise bridge equation relating the displayed Stokes constants to discontinuities. These choices matter for the Leibniz and chain rules assumed in Appendix C and for the factors of $8\pi g$ in (5.13), (5.19), and (5.30).

   The exact cusp and physical mass gap are real for positive coupling, whereas individual sectors in (5.1), (5.18), and (5.27) carry phases $e^{i\alpha\Delta}$. The authors should state the trans-series parameters and the lateral or median prescription that cancels ambiguities and yields the real observable.

They should then say whether (5.35) is an identity of formal sector coefficients, of upper lateral sums, of median Borel–Écalle sums, or of independently defined exact functions. A formal equality is valuable, but it is not automatically an equality of resummed observables.

The physical value $a = 1/4$ requires special care because different combinations of actions become commensurate, beginning with Λ_+^2 and Λ_-^6. Please explain whether resonant logarithmic sectors can occur, why none are required if that is the conclusion, and how the (δ^+, δ^-) representation combines all contributions at the same action. The cancellation of ambiguities should be displayed at least at this first mixed order. This would turn the attractive qualitative explanation around (4.12) into a controlled physical prescription.

4. **The tilted mass gap needs a more complete derivation and a clear status.** Equation (4.1) is the main new definition, yet the reduction in (4.2) to the physical O(6) gap is described as straightforward and not shown. Because the normalization, phase, and sign of this object control all subsequent relations, the authors should carry out the $a = 1/4$ reduction explicitly using the identities and quantization condition cited in the text. The square-root branch in (5.29) should be matched to the same exact definition.

   The sign ambiguity in $c_+^{(0,0)}$ in equation (3.19) is currently fixed because only one sign produces the expected physical mass gap and matches the later decomposition. The text itself says that a rigorous proof is lacking. The sign should instead be fixed from the exact resolvent definition, a normalization or positivity condition, or a controlled asymptotic limit. Otherwise all results depending on it should retain an explicit branch assumption.

   The allowed parameter range also needs correction. The manuscript states $0 \leq \alpha \leq \pi/2$, while central formulae contain $1/\sin(2\alpha)$, $1/(1-2a)$, and $x_0^+ = 1/2 - a$. Both endpoints are singular or degenerate, and the strong-coupling action disappears at $a = 1/2$. The generic analysis appears to require $0 < a < 1/2$, with endpoints treated through separate limits. Rational values, including $a = 1/4$, should be distinguished from generic nonresonant values. Finally, if no independent deformed theory is known, $m(\alpha)$ should be described as an auxiliary tilted mass-gap generating function rather than a physical mass for generic α.

5. **The foundational BES/resolvent normalization and Appendix A operator derivation must be repaired.** Equation (A.6) should presumably read $\boldsymbol{\mathcal{H}} = \sum_n |\Psi_n\rangle\langle\Psi_n|$; as printed it has a free m and is not a projector. With that correction, equation (A.11) implies either $(1 + \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{X}})\boldsymbol{\gamma} = \boldsymbol{\mathcal{H}}$ or, after the standard push-through identity, $\boldsymbol{\gamma}(1 + \boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{H}}) = \boldsymbol{\mathcal{H}}$. It does not imply the right-multiplied ordering in (A.12). The step from (A.13) to (A.14) is valid when one starts from the latter correct identity, because $\boldsymbol{\mathcal{H}}|\Psi_n\rangle = |\Psi_n\rangle$. Please rewrite the argument accordingly and verify the signs and matrix ordering in (A.15)–(A.17).

   This is not merely a typographical concern: Appendix A is invoked to derive the generalized BES equations and the central relation (3.1). The revised paper should show that the correctly ordered operator equation leads to the same integral system. It should also correct the statement before (A.17) that takes an $x \to 0$ limit when the displayed definition uses the other kernel variable in that limit.

   There is also a normalization mismatch between the main text and this appendix. From (A.2) and the orthonormality convention, the scalar matrix element of the symbol is $K_{nm}/2$, with K_{nm} defined in (1.8). Combining this with the matrix U in (A.4) does not reproduce the overall $2\cos\alpha$ multiplying the blocks in (1.6); the odd–odd block differs by a factor of four as the equations stand. The authors should state the Hilbert-space measure and reconcile (1.6), (1.8), and (A.2)–(A.4), then recheck the resolvent normalization at $a = 1/4$. Finally, (2.3) and (2.5) require a plus sign between the two terms in (2.8), while (2.8) prints a minus. This sign must be corrected or an unprinted convention explaining it must be given; the plus sign is also the one consistent with the combination used later in (3.23)–(3.24).

6. **The two trans-series bases and the claim of completeness need a precise general formulation.** Equation (5.3) gives several useful examples relating the (n, m) and (δ^+, δ^-) descriptions, but the map is not one-to-one and no general prescription is stated. Repeated labels vanish for the determinant because of the Stokes recurrences, whereas repeated weights survive in the cusp and mass gap through the ratio and square root. At the same time, the manuscript alternates between calling $\delta^\pm$ sets and multisets. These distinctions are central to the proposed Alien algebras, not just notation.

   Please define the admissible index objects, give the general action-to-(n, m) map or an algorithm for forming every aggregate coefficient, and state exactly in what sense the resulting trans-series is complete. The discussion should cover coincident total actions at rational a, not only the examples in (5.3). It should also separate the nilpotent determinant algebra (5.15) from the nonnilpotent cusp and mass-gap algebras. This would make the factors $N_j^\pm + 1$ and the ratio formulae considerably easier to verify.

7. **The finite-order evidence for the proposed algebra should be made reproducible and should include a direct observable check.** The manuscript reports hundreds of $1/g$ coefficients in some directions and computations through $\Lambda_-^{16}\Lambda_+^{16}$, but no representative data, recurrence output, or independent numerical comparison is shown. Since the all-sector claim is currently supported primarily by such checks, the evidence should be visible.

   I suggest providing machine-readable coefficient data or a compact ancillary notebook, together with several analytic mixed-sector examples beyond those already inherited from the determinant. A particularly useful test would compare an optimally truncated or resummed version of the tilted mass-gap trans-series with an independent numerical solution of its defining BES/Fredholm expression at a generic nonresonant a. At $a = 1/4$, a comparison with an independent O(6) TBA or exact mass-gap calculation should include the first mixed Λ^6 order. Large-order tests of the predicted Borel singularities and Stokes constants would directly support the proposed Alien algebra rather than only checking algebraic coefficient identities.

8. **The new results must be distinguished explicitly from the recent papers on which the manuscript builds.** Section 5.1 is largely a review of the determinant shift and Stokes structure, and Section 5.2 substantially reviews the tilted-cusp framework. The strongest original results seem to be the origin matching in Sections 3.2–3.3, the tilted mass-gap construction (4.1)/(4.3), its generic sectors, the two-scale explanation of the Dorigoni–Hatsuda breakdown, and the proposed mass-gap relations (5.30)–(5.35). The Introduction should provide an itemized comparison with the cited Boldis 2026 and Bajnok–Boldis–le Plat 2026 papers, including a precise account of what mass-gap information is already present in the latter work.

   The comparison should also engage more directly with the cited trans-series construction for conserved charges in integrable O(N) theories. A direct cross-check or an explanation of how the present Fredholm-determinant organization differs would make the significance clearer. If the generic tilted quantity has no independent physical interpretation, the title and abstract should avoid suggesting that every member of the one-parameter family is an AdS/CFT observable.

A sharper novelty statement would strengthen, rather than weaken, the genuinely interesting physical $a = 1/4$ result.

Minor comments

1. Equation (1.1) repeats D^{k_1} in the first two factors; the second exponent should presumably be k_2.

2. The Introduction states after equation (1.4) that $m_{\mathrm{O}(6)}$ is proportional to $\Lambda^2 \sim e^{-2\pi g}$. Equation (4.11) instead has $m_{\mathrm{O}(6)} \sim e^{-\pi g}$; it is $m^2_{\mathrm{O}(6)}$ that is proportional to the scale Λ^2.

3. The paragraph following equation (2.18) reverses the zeros and poles of χ_α. The $x_j^\pm$ are zeros, consistently with equations (2.8) and the later discussion, whereas the y_j label the poles.

4. Equation (3.9) separately adds a perturbative term and then sums over $n, m \geq 0$, apparently including $(0,0)$ again with an extra factor of g^2. The sum should exclude $(0,0)$, or the notation should be redefined so that the large-g scaling and the convention stated below (3.10) are unambiguous.

5. In equation (5.9), the products are written over δ_j^- and δ_j^+, which are not defined; these presumably mean δ^- and δ^+. Equation (5.12) also contains a doubled $\mathcal{I}_2$.

6. Define the Dirichlet beta function when it first appears and distinguish it from other beta-function conventions used in field theory.

7. The sentence before equation (3.19) says that one initial value determines the entire origin solution. State the asymptotic or boundary conditions that remove the remaining integration constants of the coupled second-order system (3.14).

8. The claim below equation (5.1) that all coefficient functions and Stokes constants are real should be accompanied by its precise domain and inheritance from the cited determinant result. This fact is used essentially when arguing that the Alien derivative fixes a full sector rather than only its ambiguity.

9. Correct "titled mass gap" to "tilted mass gap" in Section 6 and use O(6) consistently. The notation near the end of Section 5.4 also contains $m^{(\{(0)^{(n)}\},\{\})}$, with apparently superfluous parentheses.

10. The generic-angle relation (4.10) is presented as a new extension in Section 4, whereas Section 5.3 attributes the corresponding statement to literature focused on the physical case. Separate the established $a = 1/4$ relation from the new generic-angle derivation and state the provenance consistently.