Referee Report

Some Remarks on the Spectral Geometry of the Gribov Horizon

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Summary of the paper

The manuscript develops a local operator-theoretic description of Landau-gauge Gribov-horizon crossings. Its first objective is to separate two elliptic operators that act on adjoint scalars but have different geometric roles. The covariant Laplacian $\Delta_A = D_A^* D_A$ controls the orthogonal decomposition of the tangent space to the space of connections, while the Landau Faddeev–Popov operator $\mathcal{M}[A] = -\partial_\mu D_\mu[A]$ is the pulled-back Hessian of the orbit-norm functional. On a torus, the author removes the fixed kernel of spatially constant ghosts and defines the reduced Faddeev–Popov operator on $\mathcal{H}_0$. Theorem 2.1 then relates the negative index of this reduced Hessian to the singular parameters of the full affine focal pencil by transporting its quadratic form to $1 + sK_A$, with K_A compact and self-adjoint.

The second objective is to describe a local crossing through its critical projector P and compressed derivative $\Gamma = P\dot{\mathcal{M}}P$. Proposition 3.1 recovers concavity of the lowest reduced eigenvalue and convexity of the first Gribov region. Theorem 3.2 gives the exact identity $\Gamma = -t_*^{-1} P\mathcal{M}_0 P$ along an affine ray beginning in the positive region, so every critical branch has negative radial velocity. Section 4 uses a Feshbach–Schur reduction to obtain the residue $P\Gamma^{-1}P$ at a regular degenerate crossing. For a tangential path, a second Schur reduction on $\ker\Gamma$ determines the higher pole order, as summarized in Proposition 4.1. Section 5 discusses symmetry-protected multiplets and the expected codimension of degenerate zero blocks.

The final part contains three applications. Section 6 studies a reducible, sector-projected single-harmonic background on T^2 and checks the Feynman–Hellmann slope numerically. Section 7 decomposes an $SU(2)$ hedgehog on $\mathbb{R}^3$ into radial spin–orbit channels. For the profile in equation (99), the author constructs the positive threshold state in equation (104) in every coupled channel, derives the three threshold families in equations (107)–(109), and minimizes over all $L \geq 1$. The claimed result is the exact threshold-stability interval $-2 < g < 1$, bounded by a $J = 2$ quintet at $g = -2$ and a $J = 0$ singlet at $g = 1$. Dirichlet-box calculations are used as numerical comparisons. Section 7.4 finally proposes two representation-space matrices to illustrate splitting of the next $J = 1$ triplet.

Overall assessment

There is much that is careful and useful in this manuscript. The separation of Δ_A from $\mathcal{M}[A]$ is conceptually important, and the treatment of the fixed constant-ghost kernel avoids a common ambiguity in statements about the first nontrivial horizon. The form-domain proof of Theorem 2.1 is concise and, as an index identity, convincing. The affine-ray compression in Theorem 3.2 is particularly clean: it controls every vector in a degenerate critical block and does not require simplicity of the lowest eigenvalue. The two-stage Schur analysis also explains correctly why a Moore–Penrose inverse of the first crossing matrix cannot determine a tangential pole.

The central hedgehog calculation is likewise largely correct. Equations (93)–(96) give the expected spin–orbit reduction. Direct substitution of the state in equation (104) gives the quoted amplitude, its behavior $u_L \sim r^{L+1}$ at the origin and $u_L \sim r^{-L}$ at infinity is normalizable for every coupled channel, and the three values of c_{JL} in equation (106) exhaust $J = L-1, L, L+1$. The monotonicity used in equation (110) then places the first positive and negative thresholds at 1 and -2, respectively. The beta-function quotients in equations (117)–(119) and the Appendix are also consistent.

The present version nevertheless contains a genuine correctness error in the triplet analysis, and two central claims require stronger justification and more accurate positioning relative to prior work. In particular, equation (122) is incompatible with the real structure of the Faddeev–Popov operator; the half-line stability argument is not yet stated at the level of self-adjoint operators and closed forms; and the profile at the negative endpoint exactly reproduces a construction already given in Ref. [21], without that overlap being explained. These are substantial but correctable issues.

Recommendation: Major revision.

I would be willing to reconsider a revision that corrects the triplet representation theory, supplies a domain-level proof of the full hedgehog stability statement, and sharply distinguishes the new all-channel result from the cited earlier zero-mode construction. For a journal such as PRD or JHEP, the author should also provide a more direct demonstration that the compact crossing framework has nontrivial content beyond standard spectral and Schur reduction arguments.

Major comments

1. The reflection-odd triplet splitting in equation (122) is not a real Faddeev–Popov crossing matrix

The operator $\mathcal{M}[A]$, and hence a variation $V_B = -\partial_i \,\mathrm{ad}_{B_i}$ generated by a real transverse gauge field, preserves real adjoint-valued ghosts and is self-adjoint on the real ghost Hilbert space. Its compression to the real $J = 1$ critical multiplet must therefore be a real symmetric operator. This requirement is not respected by the proposed matrix

$$D_{\mathrm{odd}} = P_1 J_3 P_1 = \mathrm{diag}(-1, 0, +1)$$

in equation (122).

The issue is obscured by the use of the complex spherical basis $|1, m\rangle$. On the complexification of the real $J = 1$ representation, the reality involution satisfies $C|1, m\rangle = (-1)^m |1, -m\rangle$. A physical crossing matrix must commute with C, whereas $C J_3 C^{-1} = -J_3$. Equivalently, in a real Cartesian basis J_3 is, up to the conventional factor of i, the antisymmetric generator of rotations in the transverse plane; it is not a real symmetric operator. Thus the fact that equation (122) is diagonal and Hermitian in the complex helicity basis does not make it an admissible compression of a real Faddeev–Popov variation.

There is also a simple real-representation formulation. On the real $m = \pm 1$ plane, the commutant of axial $SO(2)$ consists of $aI + bK$, where K is the real antisymmetric rotation generator. Self-adjointness forces $b = 0$. Consequently a real, self-adjoint perturbation preserving axial $SO(2)$ cannot split this doublet. Breaking only the chosen axial reflection while preserving all axial rotations does not change this conclusion.

Equation (121) is compatible with the real structure, but the $\eta D_{\rm odd}$ term in equation (123) is not. The author should remove that term and revise Figure 3 and the associated statements in the Abstract, Introduction, Discussion, and Conclusion. Alternatively, the author could study a real symmetric quadrupolar operator such as a projection of $\{J_x, J_y\}$ or $J_x^2 - J_y^2$. Such a perturbation necessarily breaks axial $SO(2)$ and splits real linear polarizations rather than the two helicities. In either case, calling a pattern "allowed" should be supported by an explicit real transverse deformation $B_i^a(x)$ or by a proof that the compressed image in equation (82) contains the proposed tensor.

2. The threshold-stability interval needs a closed-form and domain-level proof

The argument following equation (104) is physically plausible and reaches the correct answer, but the present invocation of Sturm oscillation is too quick for the central claim. The radial problems are singular half-line operators, and zero lies at the bottom of their essential spectrum. The manuscript should define the self-adjoint realization of

$$H_{JL}(g) = -\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2} + c_{JL}g\phi(r)$$

on $L^2(0,\infty)$, including the boundary behavior at the origin, and state the relative-compactness or form-boundedness facts that fix the essential spectrum.

For the positive nodeless state u_* at $g = g_*(J,L)$, a short ground-state transform would make the required conclusion rigorous. On a form core one has

$$q_{g_*}[f] = \int_0^\infty u_*^2 \left| \left(\frac{f}{u_*} \right)' \right|^2 dr \geq 0,$$

with the behavior of u_* at zero and infinity controlling the boundary terms. Closing the form proves $H_{JL}(g_*) \geq 0$. Form monotonicity in the attractive direction then shows that no negative eigenvalue occurs between the free point and the threshold. This directly supplies the step that is now assigned to a general oscillation statement at the edge of the continuum.

The manuscript should then state explicitly that the (J,L) partial-wave decomposition is an exhaustive orthogonal direct sum for an adjoint scalar with color spin one. Together with the monotonicity of all three families in equations (107)–(110), this promotes channel-by-channel positivity to nonnegativity of the full Faddeev–Popov operator throughout the interval in equation (112). The uncoupled $L = 0$ channel should remain visible in this argument: it keeps the bottom of the essential spectrum at zero, so the result is never a positive-gap statement.

The finite-box comparison also deserves a short variational convergence argument. At minimum, the author should show why the zero of the Dirichlet ground-state curve converges to g_* as $R \to \infty$, rather than only quoting local Sturm–Liouville convergence. If the continuum crossing quotients in equations (117)–(119) are claimed as limits of the normalized box slopes, the uniform tail control mentioned below equation (116) must be established. If it is not established, the text should present those quotients strictly as analytic comparison values rather than as proved limits.

3. The overlap with Ref. [21] must be disclosed and the novelty isolated

The literature comparison in Section 7.1 is presently inadequate. Ref. [21], which includes the present author, studies the identical three-dimensional hedgehog $A_i^c = \epsilon_{cij}x_jh(r)$. Its $p = 3$ example gives

$$h(r) = -\frac{18\alpha r}{(r^3+\alpha)^2}, \qquad \psi(r) = \frac{1}{r^3+\alpha}.$$

At $\alpha = 1$, this is exactly equation (99) at $g = -2$, and its radial zero mode is the $L = 1$ state appearing in equation (104). The manuscript currently cites Ref. [21] only as part of the general Henyey framework and presents the profile as though the entire threshold construction began here. That presentation is not acceptable for assessing novelty.

The likely new result is not the negative-endpoint background or its zero mode, but the diagonal $\mathbf{S}\cdot\mathbf{L}$ organization of the full channel tower, the closed formula (104) for every coupled channel of this same family, and the all-channel minimization that identifies the additional positive endpoint and the interval $(-2, 1)$. The revised manuscript should say this explicitly, compare notation and normalization with Ref. [21], and explain which thresholds or representation multiplets were not previously derived. The bibliography should also discuss the directly relevant Henyey construction by R. R. Landim, L. C. Q. Vilar, O. S. Ventura, and V. E. R. Lemes, "On the zero modes of the Faddeev–Popov operator in the Landau gauge," J. Math. Phys. 55, 022901 (2014), arXiv:1212.4098.

More broadly, Theorem 2.1 is a compact Birman–Schwinger/min–max count after the form equivalence in equation (32), Theorem 3.2 follows from the compressed affine zero equation, and much of Section 4 is standard Feshbach and Schur algebra adapted to the critical ghost space. This does not make those sections unhelpful, but the author should distinguish a useful synthesis and adaptation from a new general spectral theorem. The manuscript would benefit from a specific statement of what is new relative to standard crossing-form, spectral-flow, and analytic perturbation theory.

4. No example currently realizes the full compact, irreducible crossing framework

The paper's most general claims concern a compact manifold, an irreducible connection, an isolated critical eigenvalue, and a locally defined resolvent pole. None of the examples simultaneously has these properties. The single-harmonic model of Section 6 is reducible and sector-projected, as the author correctly emphasizes. The hedgehog of Section 7 lives on $\mathbb{R}^3$, its zero mode lies at the essential-spectrum threshold, and the manuscript expressly states that the compact Laurent analysis does not transfer verbatim. The radial Dirichlet box does not specify gauge-compatible boundary conditions for the full gauge field and gauge group. Finally, Section 7.4 does not construct a transverse deformation and, in its present form, includes the inadmissible matrix discussed above.

As a result, the focal index, the sourced residue $P\Gamma^{-1}P$, and especially the twice-reduced tangential pole of Proposition 4.1 are never exhibited for an actual gauge background within the hypotheses of the principal framework. For a high-energy theory journal, this significantly weakens the claimed physical payoff. I strongly encourage the author to add one compact irreducible example that shows an isolated crossing and tests the predicted residue. An example of a tangential path would be particularly valuable because that is the least standard part of the organization in Section 4. If such an example is not available, the Abstract and Conclusions should be recast more modestly and the paper positioned as a mathematical synthesis plus an independent continuum threshold benchmark.

5. Continuity of the source is insufficient for the remainders in equations (62) and (71)

Equations (62) and (71) assume that J_t is continuous at t_* but write the remainder of the sourced inverse as $O(1)$. This does not follow. If $J_t = J_{t_*} + o(1)$, multiplication of the source difference by the pole gives $o(1)/(t - t_*)$, which need not be bounded. For example, a critical component of order $|t - t_*|^{1/2}$ produces a weaker but still divergent term.

The stated $O(1)$ formula is valid for a fixed source or under the stronger condition $J_t - J_{t_*} = O(t - t_*)$, for example local Lipschitz or C^1 dependence in the relevant Hilbert norm. With mere continuity, the appropriate remainder is only $o(|t - t_*|^{-1})$ in addition to the bounded operator remainder. Likewise, the statement that the sourced singularity vanishes "precisely" when $PJ_{t_*} = 0$ identifies the coefficient of the leading simple pole, but does not imply that the full sourced solution remains bounded for a merely continuous moving source. The assumptions and conclusions in equations (62) and (71), and in the surrounding discussion, should be corrected.

6. Morse–Bott terminology should be restricted away from the nontrivial horizon

The fixed constant-ghost kernel is correctly identified in equations (9) and (10). At an irreducible configuration in the interior of the first region, it is also reasonable to describe the global-color critical orbit as Morse–Bott, because the reduced Hessian is nondegenerate. At a nontrivial horizon, however, equation (30) itself shows that the kernel contains both the tangent directions generated by global color and an additional reduced kernel. The global-color critical orbit is then not Morse–Bott in the strict sense unless the author proves the existence of a larger critical submanifold whose tangent space is exactly the full kernel.

This affects the title of Theorem 2.1 and several summary statements, although not its algebraic index identity. The safe formulation is that the endpoint has a fixed residual-symmetry kernel together with a possible additional horizon kernel, and that equation (30) splits these two contributions. Terms such as "Morse–Bott endpoint" should either be restricted to the nondegenerate reduced case or replaced by "residual-symmetry endpoint splitting." The same qualification is needed where equation (22) is called the normal Morse–Bott Hessian at the horizon.

There is a related metric point. The space $\mathcal{H}_0$ is the L^2-orthogonal complement of constant ghost parameters. It is a convenient representative of the quotient by constants, and self-adjointness makes it invariant under $\mathcal{M}[A]$. It is not, in general, the pullback of the orbit-metric orthogonal normal to the global-color orbit, which would instead be characterized through $\langle D_A \xi, D_A c\rangle = \langle \xi, \Delta_A c\rangle = 0$. The kernel and inertia of the descended Hessian are represented correctly on $\mathcal{H}_0$, but the manuscript should not identify this particular complement literally with the orbit-metric normal space without an additional argument.

7. State the analytic setting required for the spectral wall and Schur expansions

The manuscript announces Sobolev completions in Section 2, but several later geometric statements still lack the hypotheses that make them theorems. Equations (38)–(45), for example, use the differential of the simple lowest eigenvalue and identify its zero set with a regular wall. The author should fix a Banach space of transverse connections, such as H^s with $s > d/2 + 1$, state the common operator domain, and verify that $A \mapsto \widehat{\mathcal{M}}[A]$ is a C^1 or analytic type-(A) family. The Banach implicit-function theorem would then justify the tangent-space statement in equation (45).

Similarly, Section 4 and Proposition 4.1 should state that the family is a common-domain analytic self-adjoint family, that the zero eigenspace is isolated and finite dimensional, that the Q compression remains invertible near zero, and that the displayed remainders hold in operator norm. Proposition 4.1 also needs invertibility of the full operator in a punctured neighborhood on the side where its inverse is used. These conditions are available for the compact affine Faddeev–Popov family, but writing them into the results will delimit which parts remain valid for a general analytic path.

Finally, the continuum profile has a regularity point that should be addressed. Near the origin, equation (99) gives $h(r) \sim 9gr$ and therefore $A_i^c \sim 9g\epsilon_{cij}x_j|x|$. This field is sufficiently regular for the radial form calculation, but it is not a smooth connection at the origin in the usual C^∞ sense. The author should state the regularity class being admitted in Section 7 and check that it is adequate for every geometric or Yang–Mills statement made about this background, or replace the core by a smooth profile if smoothness is intended.

Minor and technical comments

1. After equation (105), define the scaled profile explicitly. The scaling statement is correct for $h_\alpha(r) = 9gr/(r^3 + \alpha)^2$, but this expression is currently left implicit.

2. In deriving equation (102), $p = 0$ also cancels the r^4 coefficient. It is the trivial, nonnormalizable free solution and should be explicitly discarded before selecting $p = (2L + 1)/3$.

3. In Section 6, add a short reality count explaining why the $q = 1$ and $q = -1$ complex blocks together give four real critical directions. This would prevent the stated multiplicity from appearing to double-count complex conjugate modes.

4. State the topology of every $O(1)$ resolvent remainder. In Sections 4 and 7 it should be clear whether the estimate is in the operator norm on L^2, between a form domain and its dual, or only after application to a specified source.

5. For the affine Faddeev–Popov family, the fixed-P critical compression in equation (57) is exactly linear in $t - t_*$; the $O((t - t_*)^2)$ term becomes relevant only for a more general analytic path. Separating these two cases would make the relation between Sections 3 and 4 clearer.

6. The numerical information in Section 6 and Section 7.2 should be made reproducible. Please give the root-finding procedure and tolerance, the precise finite-difference matrix at the first and last grid points, and, preferably, a short code or machine-readable table. The quoted convergence is plausible, but it should not depend on undocumented implementation choices.

7. The proof of Theorem 2.1 should state explicitly that the order -1 operator in equation (28) is bounded and compact on L^2 by elliptic regularity and the compact Sobolev embedding, and that the sum in equation (29) is finite because a compact operator has only finitely many eigenvalues below -1.

8. In the discussion of the Dirichlet box, distinguish convergence of the root $g_{*,R}$, convergence of normalized eigenfunctions on compact sets, and convergence of the normalized crossing quotient. These are related but not identical statements.

9. The claim after equation (82) about expected codimension is appropriately qualified by transversality. It would be useful to connect this qualification directly to the failed triplet construction: representation-space matrices alone do not show that a matrix lies in the image of the local gauge-field perturbation map.

10. The Abstract currently advertises "symmetry-allowed splitting matrices." This must be revised even if the invalid odd matrix is simply deleted, because only one allowed pattern has actually been displayed and no gauge-field deformation realizing its coefficient has been constructed.

11. A few results that are presented as geometric statements are, in the proofs, statements about pulled-back quadratic forms on ghost parameters. The distinction made below equation (23) is good and should be maintained consistently in the Introduction and Conclusion.

Conclusion

The manuscript contains a sound and potentially useful core. In particular, the reduced constant-ghost formulation, the affine-ray block inequality, and the complete spin–orbit minimization are worthwhile. The exact channel tower turns a previously known hedgehog zero mode into a stronger all-channel benchmark and appears to be the paper's most interesting new result. However, the real-structure error in equation (122), the incomplete half-line operator argument, and the current treatment of prior work prevent publication in the present form. Correcting these points, and either supplying a compact irreducible example or narrowing the claims of the general framework, would substantially improve both the correctness and the significance of the paper.