Referee Report

"Worldline representation of multiloop amplitudes in quantum electrodynamics"

arXiv:2607.13321

Recommendation: Major revision.

Summary and overall assessment

The manuscript is a conference-style review of the worldline approach to perturbative QED, with emphasis on the possibility of integrating globally over photon insertion points rather than decomposing the integration domain into ordered sectors. Sections 1–4 review the proper-time representation, scalar and spinor one-loop master formulas, their relation to second-order fermion Feynman rules, and constant background fields. Sections 5 and 6 introduce circular-integration identities and multiloop Green functions. The principal research content is then presented in Sections 7–9: three families of worldline master integrals, a compact representation and low-energy expansion for the two-loop scalar-QED vacuum polarization, and a re-evaluation of a three-loop vacuum integral relevant to the ϕ^4 beta function. Section 10 briefly reviews the open-fermion-line formalism, and Section 11 describes possible future applications.

The organizing idea is interesting and potentially useful. In particular, a reliable global-integration algorithm for the two-loop scalar-QED polarization would be valuable, both as a calculation in its own right and as a test bed for more difficult spinor-QED and $g-2$ applications. The presentation also succeeds in conveying why the worldline organization can expose cancellations that are spread over many conventional diagrams.

Several parts of the technical framework are convincing. Equations (5) and (7) are mutually consistent once the bosonic zero mode is removed; the identity in equation (11) and the $n = 1$ limits of the chain formulas in equation (13) check directly; equation (15) is the standard one-insertion Green function; and equations (16)–(17) reproduce the known self-dual two-loop Euler–Heisenberg results. The proper-time scaling from equation (29) to equation (31) is dimensionally consistent. Moreover, expanding equation (35) gives equation (37), and equations (38)–(39), if applicable, yield

$$I_{\text{reg}} = 12\zeta(3) - 4\zeta(2) - 4,$$

which is exactly the sum of the planar and nonplanar values in equation (36).

The central new claims, however, are not yet presented at a level that permits independent verification. The domains and even some definitions of the master integrals are incomplete; the main three-loop application uses equation (19) outside its stated domain; the reduction of the scalar-QED integrand to the new master families is omitted; and the renormalization statement following equation (34) does not distinguish subdivergences from the overall photon-field divergence. The manuscript also needs a substantially sharper account of novelty, because a large portion of Sections 5, 7, and 9 already appeared in a closely related 2026 proceedings article, while older scalar-QED spectral results constrain the claimed novelty of the full vacuum-polarization function. These matters are substantive rather than cosmetic. I therefore recommend major revision before publication. With the revisions below, the work could become a useful LL2026 proceedings contribution; in its present form it does not meet the self-contained correctness and novelty standards of a standalone PRD/JHEP article.

Major comments

1. **The incremental contribution and relation to prior literature must be stated precisely.**

   Much of the manuscript is review material, which is appropriate for conference proceedings, but the reader is not told clearly which formulas are new here. In particular, V. M. Banda Guzmán, J. P. Edwards, C. Moctezuma Mata Zamora, L. A. Rodríguez Chacón, and C. Schubert, "Integration techniques for worldline integrals," arXiv:2603.18442, already presented equation (18) and the ${}_3F_2$ result (19), the family in equations (22)–(28), the chain identities, and essentially the complete calculation in equations (35)–(39). The correspondence is direct: current equations (18)–(19), (22)–(28), and (35)–(39) match equations (44)–(45), (38)–(43), and (46)–(50), respectively, of that earlier proceedings article, up to relabeling. That work is not cited. By contrast, the family in equations (20)–(21) and the scalar-QED representation and coefficients in equations (29)–(34) appear to be the genuine incremental content of the present paper.

   The literature statement after equation (34) also needs refinement. Schwinger's scalar-QED result in *Particles, Sources, and Fields*, Vol. II, and the later analyses by M. Drees and K.-I. Hikasa, Phys. Lett. B **252** (1990) 127, and A. Arhrib, M. Capdequi–Peyranère, and A. Djouadi, Phys. Rev. D **52** (1995) 1404, arXiv:hep-ph/9412382, give the inclusive order-α scalar-pair spectral correction. The last paper states explicitly below its equation (3.11) that the equal-mass result agrees with scalar QED after correction of a misprint in Schwinger's formula. These works do not tabulate precisely the same bare, dimensionally regulated Euclidean function used here, but their absorptive part determines the renormalized correlator up to subtraction by a dispersion relation. Thus the defensible novelty is a compact off-shell worldline representation and a global method for its low-energy moments, not the first physical momentum dependence of the scalar-QED correlator.

   The abstract, introduction, Section 8, and Outlook should be rewritten to identify this incremental result, cite arXiv:2603.18442 and the scalar spectral literature, and compare at least several renormalized moments with moments obtained from the known spectral function. Relevant two-loop scalar-QED ultraviolet analyses should also be discussed. This would make the novelty both narrower and stronger.

2. **The master-integral families in Section 7 are not self-contained.**

   The quantity C is used in equation (18) without definition. With the index assignment used later, it appears to mean

   $$C = G_{13} - G_{14} - G_{23} + G_{24},$$

   or the corresponding $1, 2, a, b$ expression, but this must be stated before the first integral. Likewise, the shorthand $\int_{12}$ and $\int_{ab}$ in equations (20), (22), (32), and (33) is never defined. All measures and integration ranges should be explicit, including whether one insertion point has been fixed by translation invariance and, if so, which compensating factor has been included.

   The integer domains also need a systematic statement. Equation (20) uses $m, k \in \mathbb{N}$, even though $k = 0$ is needed in later expansions and conventions for $\mathbb{N}$ differ. Equation (24) contains $(k-1)!$, so equations (22)–(28) require at least $1 \le k \le m$, not merely $k \le m$. At $j = 0$, equation (23) invokes d_{-1}^{m-1}, while equation (28) invokes c_{-1}^{m-1}; the presumed zero conventions must be supplied. Empty inner sums in equation (23) need a convention as well. Since the closed form is written using $\dot{G}_{12}$, its symmetry under $1 \leftrightarrow 2$, its coincident-point limit, and the logarithm branches should be explained.

   For journal-level reproducibility, the authors should provide derivations or an archival proof of equations (19), (21), and (23), plus a table of nontrivial low-index checks against direct sectorwise or numerical integration. The present description "simple/more difficult/difficult" is not a substitute for hypotheses and proof.

3. **The principal application of equation (19) lies outside its stated domain.**

   Equation (18) states $k, l \le m$, and equation (19) is presented under that restriction. Section 9 then applies equation (19) to
   $$I_{nn}^{n-1},$$
   for which $k = l = n = m + 1$, in order to obtain equation (38). The calculation is therefore not justified by the stated result. The authors must either prove equation (19) on an enlarged convergence domain, specify and justify an analytic continuation or boundary extension, or derive equation (38) separately. The fact that equations (38)–(39) lead to the correct sum of the two values in equation (36) is a useful check, but it does not repair this logical gap.

   A related endpoint problem occurs in Section 8. Expanding Δ^n as in equation (33) produces a $k = n$ term with C^0, corresponding to $m = n - k = 0$, whereas equation (21) assumes $m \ge 1$. The $n = 0$ coefficient is entirely of this type. If these terms are handled by a separate identity, contact term, or limiting prescription, that step must be displayed rather than subsumed under the statement that equation (21) can be used.

4. **The derivation of the scalar-QED vacuum polarization is the central missing bridge.**

   Before equation (29), the paper should define the tensor convention, for example
   $$\Pi_{\mu\nu}(k) = \left(k_\mu k_\nu - \delta_{\mu\nu} k^2\right)\Pi(k^2),$$
   or its actual sign convention, and state whether the momentum-conserving delta function is stripped. The internal-photon gauge and the symmetry factor associated with sewing the one-loop four-photon amplitude should also be given. A short derivation should show how the scalar seagull diagrams and their multiplicities are represented.

   The jump from equations (29)–(30) to equations (32)–(33) is presently too large. The numerator I contains $\ddot{G}_{ab}$, derivatives with respect to several insertion points, and terms written with C^{-2}. The authors should show explicitly how integration by parts and Green-function identities reduce every term to the Section 7 master families. This requires accounting for delta-function contact terms from $\ddot{G}$, boundary contributions across the nonsmooth ordering loci, the cancellation of the apparently singular C^{-2} factors, and the endpoint cases discussed above. It would also be valuable to display the Ward-identity check, either at the integrand level or after integration.

   The dimensionless rescaling must be made explicit. Equation (32) introduces $\widehat{T}$ and u_i but continues to use γ_{ab}, Δ, I, and the derivative symbols from their dimensionful definitions. Define, for example, the rescaled $\widehat{\gamma}_{ab} = (\widehat{T} + G_{ab})^{-1}$ and the corresponding rescaled numerator. As written, the reader cannot reconstruct all powers of T or reproduce the coefficients in equation (34).

5. **Renormalization and validation of equations (31)–(34) are insufficient.**

   The parameter ϵ appearing in equation (34) is never defined. The paper should state whether $D = 4 - 2\epsilon$ or $D = 4 - \epsilon$, include the dimensional-regularization scale and its associated factors, and specify whether e and m in equation (29) are bare or renormalized. The statement that the displayed poles are "still to be removed by mass renormalization" is incomplete. Scalar-mass renormalization removes the one-loop subdivergence, whereas the two-loop polarization also has an overall photon-field/charge divergence. This distinction is especially important for c_0, for which the factor $\Gamma(4 - D)$ in equation (31) supplies an additional overall pole.

   The authors should give the mass-counterterm insertion explicitly, state the charge and mass schemes, and present finite renormalized moments. Those moments should be compared with a dispersion calculation from the known scalar spectral density. Agreement with an unpublished independent calculation is encouraging, but it is not an adequate archival check. At minimum, the paper should document what was compared, in which convention and scheme, and supply analytic or high-precision numerical checks. Machine-readable expressions or a small ancillary notebook would considerably strengthen the result.

6. **The introductory derivation needs correction, and the spinor master formula has a broken equation reference.**

   Section 2 says that inserting the plane-wave background (2) makes the path integral (1) Gaussian. Substitution alone leaves vertices of the form

   $$\int d\tau\, \dot{x}\cdot\varepsilon_i\, e^{ik_i\cdot x(\tau)},$$

   which are not quadratic in x. The Gaussian step occurs after expanding to fixed photon number, writing the vertex insertions in a source form, taking the free worldline expectation value, and projecting onto the term linear in every polarization. This logical sequence should be stated.

   Equation (5) also writes a correlator for the full periodic coordinate $x(\tau)$, although its translation zero mode prevents inversion of the kinetic operator. The paper should split $x(\tau) = x_0 + y(\tau)$ with $\int_0^T y(\tau)d\tau = 0$, use y in the correlator, and explain that the x_0 integral produces

   $$(2\pi)^D \delta^{(D)}\left(\sum_i k_i\right).$$

   Equation (6) should either include this factor or define a stripped amplitude.

   Finally, the spinor superfield master formula displayed after equation (7) is visibly unnumbered, because its last line is suppressed with `\nonumber`, yet its source label resolves to (8). The next display, the second-order identity in Section 3, is the equation actually printed as (8). Thus every later reference to the spinor master formula as equation (8) points ambiguously to the wrong display, and the PDF contains duplicate equation anchors. The spinor formula needs its own number and a clean label, followed by consistent renumbering. The Grassmann measure normalization and the four-component origin of the prefactor -2 should also be stated.

7. **The multiloop sewing claims and the Section 9 interpretation need correction.**

   Section 6 first says correctly that one inserted photon propagator produces the two-loop Green function (15), but then describes l inserted propagators $a_1b_1, \dots, a_lb_l$ as an "l-loop photon propagator." A matter loop with l internal photon insertions has $l + 1$ loops; equivalently, an L-loop quenched polarization has $L - 1$ inserted photons. The notation should be corrected, and the quantities $C_{a_ib_ia_jb_j}$ and the associated determinant matrix and measure should be defined.

   The objects in Figures 3 and 4 are more precisely 1PI vacuum-polarization insertions, not full photon propagators. A schematic sewing formula with its factor of $1/2$ would both fix this terminology and substantiate the assertion that different topologies appear with the correct multiplicities. The analogous three-loop construction should state the pair-exchange symmetry factors.

   Section 9 is entitled "Three-loop β-function coefficients," but it never gives a coupling convention, diagram symmetry factors, counterterm combination, or a coefficient of $\beta(g)$. Because all external momenta are set to zero, the authors must also specify the mass regulator or infrared-rearrangement prescription used to distinguish ultraviolet poles from induced infrared poles. The section should either complete this extraction or be retitled as a check of a three-loop vacuum master integral.

8. **Section 10 and the paper's forward-looking scope require sharper conventions.**

   For equation (40), define D_μ, the Euclidean metric and gamma-matrix algebra, and the operator factorization that fixes the spin-term sign. State that A and F in the path integral are evaluated at

   $$x(\tau) = x + \frac{\tau}{T}(x' - x) + q(\tau),$$

   and whether $(4\pi T)^{-D/2}$ is included in the normalization of Dq. Equation (42) is a four-dimensional even-Clifford decomposition; in general D there are additional antisymmetrized gamma products, while γ_5 requires a prescription. The text must therefore specialize this statement to $D = 4$ and give the on-shell relations that are said to reduce A_N and C_N to $B_{N\alpha\beta}$.

   The abstract names spinor-QED vacuum polarization and the electron anomalous magnetic moment as examples of actual multiloop calculations, whereas Section 10 describes the relevant code and $g-2$ application as future work. Likewise, the Outlook says that the present formulas "solve" two-loop QED vacuum polarization, while the manuscript actually demonstrates a bare scalar-QED representation and several low-energy coefficients. These claims should be narrowed unless a renormalized full-momentum scalar result and the spinor extension are supplied.

Minor comments

1. In the constant-field formulas, state that the trigonometric functions are matrix functions and the determinants run over Lorentz indices. In the zero-field limit, the displayed $\mathcal{G}_B$ tends to $G - T/6$ rather than literally to G; the additive constant is harmless for momentum-conserving amplitudes, but the convention should be noted.

2. In the discussion of equations (16)–(17), explicitly impose the Euclidean self-duality or anti-self-duality condition $F = \pm\widetilde{F}$. The condition $F_{\mu\nu}F^{\nu\lambda} = -f^2\delta_\mu^\lambda$ alone does not state the orientation choice.

3. The coefficient λ_{ij}^2 in the general exponential in Section 5 is undefined. If it denotes $k_i\cdot k_j$, use that notation; otherwise define it. In equation (14), the k_i should be called nonnegative integer powers, not "integer coefficients."

4. Equation (19) opens its final pair of terms with a left brace but closes with a right bracket. Use matching delimiters.

5. Equations (32) and (33) both carry the same source label, producing a multiply-defined-label warning. The $c_0, \ldots, c_4$ display also creates a stray empty equation number (34). The equation environments should be rebuilt so that every displayed number and hyperlink target is unique.

6. The bibliography contains several concrete errors: arXiv:2106.1207 in Ref. [17] should be arXiv:2106.12071; arXiv:2407.07388 in Ref. [20] should be arXiv:2407.07383; and the journal volume in Ref. [10] should be Phys. Lett. B **822**, not **22**.

7. The embedded PDF files for Figures 7 and 8 retain hidden surrounding page text outside the visible crop. They should be properly cropped or redrawn so that text extraction and accessibility output contain only the intended diagrams.

8. State the coincident-point convention for $\dot{G}$ and the periodic delta function in $\ddot{G}$, particularly when contact terms lie on boundaries between ordered sectors.

9. Explicitly connect the diagonally subtracted form of equation (15) to $\Delta = G_{12} - (\gamma_{ab}/4)C^2$ in Section 8.

10. It would improve Section 9 to state the reconstructed value $I_{\mathrm{reg}} = 12\zeta(3) - 4\zeta(2) - 4$ after equation (39), making the successful check immediately visible.

11. Correct "alternative is to performing," "internal popagators," and "with a view on applications." Use either "polarisation" or "polarization" consistently.

12. The first-person singular "Here I review" is awkward in a five-author paper unless the manuscript explicitly explains that it is the speaker's proceedings contribution.

Recommendation

The worldline organization and the new scalar-QED low-energy machinery are sufficiently interesting to merit a revised presentation. Publication should wait, however, until the authors (i) identify the genuinely new contribution and cite the closely overlapping proceedings article, (ii) correct and prove the domains of the master formulas, (iii) supply the reduction and renormalization needed to reproduce equations (29)–(34), and (iv) correct the equation-reference and multiloop-counting problems. I recommend **major revision**. A successful revision along these lines would be suitable as a substantive conference proceedings contribution; a standalone PRD/JHEP treatment would additionally require a complete renormalized result or a comparably comprehensive derivation and benchmark package.