# Referee Report

"Kink collisions in a two-dimensional gravity model"

## Summary of the manuscript

The manuscript studies kink–antikink scattering in the Mann–Morsink–Sikkema–Steele (MMSS) model of two-dimensional dilaton gravity coupled to a real scalar. The authors restrict to the consistent sector in which the dilaton is twice the conformal factor, introduce the exactly solvable asymmetric $\phi^4$-type kink of equations (5)–(11), and evolve a superposed kink–antikink pair with the dynamical equations (15) and (16), while monitoring the momentum and Hamiltonian constraints (17) and (18). The exterior of the pair is in the $\phi = -1$ AdS vacuum, whereas the region between the defects is initially near the $\phi = +1$ Minkowski vacuum.

The principal numerical claim in Section 3.1 is that increasing the parameter $\kappa$ shifts the multi-bounce windows to larger initial velocity, narrows those windows, and mildly raises the velocity above which prompt escape occurs. Section 3.2 supplements the collision evolutions with a spectral calculation for an isolated gravitating kink. The effective stability operator has an open channel on the AdS side and a closed channel on the Minkowski side. At $\kappa = 0.1$ the authors report the pole

$$d\omega_{\mathrm{res}} = 1.70583 + 7.05 \times 10^{-4} i,$$

with $Q \simeq 1.2 \times 10^3$, and propose this long-lived excitation as a possible temporary energy reservoir in weak-gravity collisions. Section 3.3 describes the accompanying geometry: the conformal factor decreases in the central region after collision, while the Ricci scalar exhibits finite transient peaks in the evolutions shown in Figures 5 and 6.

The problem is interesting. A fully backreacting collision of self-gravitating kinks in this particular two-dimensional theory appears to be a useful and reasonably novel calculation, and the asymmetric open-channel resonance could be a worthwhile specific result. Several parts of the formal analysis are also sound. In particular, equation (15) implies $R = \kappa V$ with the manuscript's curvature convention; the factorization in equation (26) indeed gives $-\partial_z^2 + f''/f$; the outgoing-wave sign in equation (27) is consistent with the convention $e^{i\omega t}$; and the value of $Q$ in equation (30) follows from the quoted pole.

In its present form, however, the paper does not yet establish its main physical attribution or the quantitative numerical claims. Most importantly, varying $\kappa$ changes the scalar potential in equation (5) at the same time as it changes the gravitational coupling. The collision trends therefore cannot currently be attributed to gravity alone. In addition, the velocity scan is too coarse to resolve the narrowing it claims, the initial data violate the constraints without a convergent Hamiltonian residual, and no convergence study is given for the evolved scattering observables. The resonance calculation and the proposed connection to the nonlinear windows also require substantially more documentation and evidence.

## Recommendation

I recommend **major revision**. The manuscript contains a potentially publishable numerical study, but the revisions required are substantive rather than cosmetic. Publication in a journal such as

PRD or JHEP would require the authors to disentangle the direct $\kappa$-dependence of the matter theory from gravitational backreaction, resolve and quantify the scattering windows, establish robustness against constraint and discretization errors, document the resonance pole at the precision quoted, and replace coordinate-dependent geometric interpretations by appropriately defined observables. The nonlinear collision calculation would then have a clear and useful place in the literature.

## Assessment of correctness and logical structure

The manuscript follows a natural progression from the exact static solution to the constrained evolution system, scattering outcomes, the isolated-kink spectrum, and spacetime diagnostics. Equations (1)–(18) give a compact formulation of the model, and the special choice $\varphi = 2A$ is consistent with equation (13) if both this relation and its time derivative hold initially. The linear stability argument in Section 3.2 is also coherent: the factorization in equation (26), the zero left threshold, and the non-normalizability of the formal zero mode support the absence of an ordinary discrete shape mode. A quasi-bound state trapped by the barrier but leaking into the left channel is physically plausible.

The weak point in the logical chain lies between these formal observations and the main conclusions. Figure 3 is used to infer systematic motion and narrowing of fractal windows, although the scan does not numerically resolve their boundaries. The isolated-kink pole in equation (29) is then offered as a possible mechanism, but the nonlinear evolution is neither projected onto this mode nor fitted to its ringdown frequency. Finally, a decrease of $e^{A}$ at fixed conformal coordinate is interpreted as a lasting geometrical contraction even though the observer congruence and the interval are coordinate-selected, and the collision also converts the central region from one vacuum geometry to the other. These missing links should be supplied or the claims should be narrowed accordingly.

## Novelty and significance

The static kink, superpotential construction, and much of the stability framework are inherited from earlier work, so the main novelty is not equations (5)–(26) by themselves. The likely new content is the fully coupled nonlinear evolution governed by equations (15)–(18), the detailed scattering map, and the particular resonance pole in equation (29). Those results could be of interest to researchers working on gravitating solitons, nonlinear scattering, and two-dimensional dilaton gravity.

The literature discussion nevertheless needs correction. F. C. E. Lima and C. A. S. Almeida, "Aspects of kink-like structures in 2D dilaton gravity," arXiv:2205.11570, use the same MMSS setting and discuss velocity-dependent kink–antikink scattering. Their geometry is prescribed rather than fully backreacting, so this does not remove the more specific novelty of the present calculation, but it is too closely related to omit. The manuscript should state precisely how its dynamically evolved geometry advances beyond that work.

The proposed physical role of a leaking mode should likewise be placed in the established quasinormal-mode collision literature. Dorey and Romańczukiewicz, arXiv:1712.10235, already showed that a long-lived mode with leakage can store collision energy and that its decay can move, narrow, or close resonance windows and change the critical velocity. The more recent nonlinear/spectral comparison in arXiv:2510.05311 is also directly relevant. The new point here would be the realization of that mechanism in an MMSS gravitating kink, not the mechanism in general. Finally, the perturbation operator in equations (23)–(26) should cite the direct derivation in Y. Zhong, “Normal modes for two-dimensional gravitating kinks,” arXiv:2112.08683.

## Major comments

1. **The variation of $\kappa$ does not isolate a gravitational effect.**

   Equation (5) contains $\kappa$ explicitly in the matter potential. Consequently, changing $\kappa$ modifies the force term $dV/d\phi$ in equation (16) directly, in addition to changing $A$ and the coupling of matter to the geometry. This is not a parametrically smaller effect. For example,

   $$V(+1) = 0, \qquad V(-1) = -\frac{2\kappa}{9d^2}.$$

   Thus increasing $\kappa$ changes the vacuum-energy difference between the initial Minkowski phase inside the pair and the AdS phase outside it. It also changes the nonlinear scalar interaction and the pressure favoring one vacuum, even though the tuned static profile in equation (6) retains its simple $\tanh(x/d)$ form.

   The statements in Section 3.1, the abstract, and the conclusions that “gravity” shifts and narrows the windows therefore do not follow from the comparison shown. The authors should introduce independent parameters for gravitational coupling and for the second term in equation (5), or perform suitable controls in which the same matter potential is evolved with and without backreaction. If this separation is not feasible, the conclusion must be reformulated as a trend within the particular self-consistent one-parameter family defined by equations (1) and (5), without assigning it uniquely to gravity. A $\kappa = 0$ run with the same code, initial separation, velocity convention, grid, and outcome diagnostic is essential in either case; comparison only to window intervals quoted from another numerical setup is not an adequate code or physics control.

   This point also affects the interpretation of Figures 5 and 6. After annihilation, the central field changes from the $\phi = +1$ vacuum toward the $\phi = -1$ vacuum, so part of the changing geometry is the expected replacement of a locally Minkowski phase by an AdS phase whose vacuum energy itself depends on $\kappa$. A vacuum or domain-wall control is needed before the change can be described as a cumulative dynamical imprint distinct from this phase conversion.

2. **The velocity scan does not quantitatively resolve the claimed resonance-window trends.**

   Section 3.1 scans in steps of $\Delta v = 0.002$, while the conclusion concerns windows that become progressively narrower and eventually difficult to see. The third flat-space interval quoted in the text has width only about 0.0024, comparable to one scan step, and the claimed gravitationally narrowed windows may be smaller still. A grid that does not resolve a window cannot establish its width, its displacement, or its absence. Figure 3 provides a useful survey, but not the quantitative evidence required for the principal conclusion.

   The authors should refine each candidate interval adaptively or by bisection, state an operational definition of a collision and of escape, assign a bounce number, and give a table of lower and upper endpoints with numerical uncertainties. The same table should quote the critical escape velocity $v_c(\kappa)$ rather than only saying that it increases mildly. Outcome classifications must be checked under longer evolution because a delayed escape can look like a bion over a finite time window.

There is also an internal reproducibility issue: velocities 0.215, 0.235, and 0.245 are not members of the stated grid $0.002, 0.004, \ldots, 0.300$. Figure 5 likewise uses $v = 0.235$. If additional fine scans or individual runs were performed, their sampling and selection criteria must be reported. The paper should not use isolated representative points as substitutes for the actual interval boundaries.

3. **Constraint-violating initial data require a controlled robustness analysis.**

   The superposition in equations (20) and (21) is not a solution of the constraints (17) and (18). Section 2.3 further states that the Hamiltonian residual does not display clean convergence as the number of points is increased. An absolute residual of order $10^{-6}$ is encouraging but is not, by itself, a continuum error estimate. In a fractal scattering problem, small changes in the initial data can change the bounce number near a window boundary; the relevant question is whether the reported intervals and critical velocities are stable as the constraint error is reduced.

   The preferred solution is to construct constraint-satisfying initial data after explicitly fixing the residual gauge freedom. A singular Newton matrix generally signals a gauge or boundary null direction that should be diagnosed, not a reason to stop the constraint solve. If the authors retain the approximate superposition, they must distinguish its continuum superposition error from coordinate-interpolation and finite-difference errors, provide normalized constraint residuals, and vary the metric data within the observed residual to demonstrate stability of every quantitative scattering claim. The physical results should also be compared for several values of $z_0$. The fact that the residual did not decrease when the separation was enlarged does not show that the collision map itself is independent of the separation.

   The caption of Figure 1 states that the initial violations propagate with growing amplitude. The manuscript should therefore show the constraint norm in the collision region as a function of time for representative bion, window, prompt-escape, and strongest-coupling runs, rather than only a few spatial snapshots.

4. **Convergence of the nonlinear evolutions and of the derived observables is not demonstrated.**

   Fourth-order spatial stencils do not guarantee fourth-order convergence of a calculation that also uses numerical coordinate inversion, interpolation, adaptive time integration, approximate initial data, and artificial boundaries. The production values of $L$, $N$, interpolation order, boundary stencil, time tolerances, and maximum allowed step are not given. Nor is there a convergence test for the quantities that enter the conclusions: field profiles, collision times, bounce counts, window endpoints, $v_c$, $\min e^A$, or $\max |R|$.

   At least three spatial resolutions and more than one temporal tolerance should be compared for cases representing a bion, a point near a window edge, multi-bounce escape, prompt escape, and the largest $\kappa$. The paper should report observed convergence orders where a smooth convergence regime exists, and direct outcome robustness where the fractal boundary prevents a simple norm comparison. Reproduction of standard flat-space $\phi^4$ scattering at $\kappa = 0$ with the same implementation would provide an especially valuable validation.

   The exact implementation and version of `ode78` should be identified, together with its error norm and tolerances. The manuscript calls it a built-in MATLAB solver, but this name has been used by several implementations; readers need enough information to reproduce the evolution.

5. **The resonance pole is underdocumented relative to its quoted accuracy.**

Equations (23)–(28) define a plausible resonance problem, and the value in equation (29) is consistent with the barrier and asymmetric thresholds displayed in Figure 4. The manuscript nevertheless supplies no algorithm for finding the complex pole. It does not state whether the calculation uses shooting, matching, a spectral discretization, or analytic continuation; how the exponentially growing outgoing solution is handled numerically; what the left and right cutoffs and grid spacings are; or what complex residual remains at the root. The assertion of stability at five cutoffs is not accompanied by the five values. This information is necessary before seven digits in $\mathrm{Re}(d\omega)$ and a width of order $10^{-4}$ can be assessed.

The revision should give the analytic limits of $V_{\text{eff}}$ on both sides, the gauge-invariant perturbation variable represented by $\delta\phi$, the numerical boundary conditions at each finite endpoint, and a convergence table in cutoff, resolution, and solver tolerance. The table should list $\mathrm{Re}\,\omega$, $\mathrm{Im}\,\omega$, $Q$, and the root residual. For $\kappa = 0.3$ and $0.5$, the searched region and the sensitivity threshold behind "no comparably isolated resonance" must be specified. A time-domain perturbation followed by a ringdown fit would provide a strong independent check.

There is a specific asymptotic issue that the convergence analysis must address. On the left, $f$ has a power-law tail and the effective potential approaches zero as an inverse square, schematically $V_{\text{eff}} \sim p(p+1)/(z-z_*)^2$ with $p = 6/\kappa$. Thus $p = 60$ at $\kappa = 0.1$, and the approach to the plane-wave boundary condition in equation (27) is enhanced by a large coefficient. Cutoffs $|z|/d = 70$–$140$ are not automatically in a regime where simply setting $\psi' = i\omega\psi$ is accurate enough to determine an imaginary part of order $7 \times 10^{-4}$. The authors should state whether they use an analytic inverse-square-tail solution, higher-order asymptotic matching, or a bare plane wave, and propagate the resulting boundary error into the quoted digits.

The small-$\kappa$ continuation is physically important. The real part of the reported frequency lies near the flat $\phi^4$ shape-mode frequency $\sqrt{3}/d$. Showing the pole trajectory and $Q$ as $\kappa \to 0^+$ would clarify whether the open-channel state is continuously inherited from the flat-space bound mode and would make the result considerably more informative.

6. **The connection between the isolated-kink pole and the nonlinear bounce windows remains conjectural.**

The caution in the final paragraph of Section 3.2 is appropriate, but the abstract and conclusions still give the resonance a prominent explanatory role. At present Figure 4 only proves the existence of a linear resonance of a single static kink. It does not show that a collision excites this profile, that the stored energy is returned at the observed escape time, or that the disappearance of resolved windows at larger $\kappa$ follows from the pole's motion.

The authors should project the post-impact perturbation onto a suitably normalized resonant profile, or fit a local ringdown to the real and imaginary parts of equation (29). They should compare the inter-bounce intervals with the phase-accumulation condition based on $\mathrm{Re}\,\omega$, and compare the decay time with the duration of the relevant collision sequences. If this cannot be done, the spectral result should be presented as a separate, interesting property of the isolated gravitating kink rather than as evidence for the mechanism of Figure 3. The existing quasinormal-mode collision literature cited above provides appropriate tests and context.

7. **The claimed local contraction is tied to a coordinate-selected congruence.**

Equation (31) is algebraically correct for $u^a = (e^{-A}, 0)$, but these "static observers" are selected by the conformal coordinates and are generally accelerated. Residual conformal transformations change $A$, fixed-$z$ intervals, the constant-$t$ slicing, and the congruence. Therefore, a decrease of $e^A$ over a fixed coordinate interval is not by itself an invariant statement that the physical system has contracted. Calling equation (31) a scalar does not remove the dependence on how the congruence was chosen.

The geometrical response should be characterized using observers or endpoints defined by the matter or geometry: for example, the proper separation of invariantly located kink centers while they exist, the expansion of a specified geodesic congruence, dilaton-defined worldlines, curvature extrema, or an appropriate asymptotic mass/charge. The manuscript should also separate this response from the change of vacuum discussed in comment 1. If no invariant diagnostic is added, the claims in the abstract and conclusions should be restricted to the behavior of the conformal factor and of the chosen static-observer congruence in the adopted gauge.

8. **The singularity statement needs numerical and geometrical qualification.**

   In two dimensions $R_{abcd}R^{abcd} = R^2$, and the field equation gives $R = \kappa V$ in the sector evolved here. Monitoring $R$ is therefore a sensible curvature test. Finite sampled values over a finite grid and time interval, however, do not establish geodesic completeness or exclude a later blow-up, a degeneration of the conformal factor, a dilaton singularity, or loss of differentiability. This is particularly relevant because the potential in equation (5) behaves as $-\kappa\phi^6/(72d^2)$ for large $|\phi|$ and is unbounded below for every $\kappa > 0$.

   The authors should tabulate the resolution dependence of $\max|R|$ and $\min e^A$, monitor relevant derivative norms and $\max|\phi|$, and state the time and parameter domain over which the claim is made. They should demonstrate that all trajectories shown remain in the metastable field range. If a stronger completeness statement is intended, affine or proper-time reach along appropriate causal geodesics must also be analyzed. Otherwise the accurate conclusion is the narrower one: no curvature divergence was resolved during the converged portion of the reported simulations.

   The curvature plots themselves require clarification. From equations (5) and (15), the exterior AdS vacuum has $R = \kappa V(-1) = -2\kappa^2/(9d^2)$, which is negative. The curvature panels in Figures 5 and 6 appear to have axes beginning at zero and display no negative exterior plateau, while their captions call the plotted quantity $R$. The authors must state whether these panels show $R$, $|R|$, a vacuum-subtracted curvature, or a clipped/rescaled quantity. Signed, quantitatively labeled curvature plots are needed; otherwise the figures do not directly display the invariant asserted in the text.

   The comparison in Section 3.3 should also recognize that singularity formation is model-dependent rather than a simple distinction between two and higher dimensions. Two-dimensional dilaton models with colliding or self-gravitating solitons have produced black-hole or singular solutions; relevant examples include arXiv:hep-th/9501045, hep-th/9504109, hep-th/0003133, and arXiv:2205.15748. Discussing why the MMSS dynamics differs would sharpen the physical significance.

9. **Boundary causality and the available observation time must be reconciled with the figures.**

   Section 2.4 states that results are restricted to $|z| < L - t$ so that boundary-generated violations cannot enter. Figure 1 reports a domain $z/d \in [-150, 150]$, while Figures 3, 5, and 6 extend to $t/d = 150$; Figures 5 and 6 display rectangular regions extending to approximately $|z/d| = 10$. If the production value is $L/d = 150$, the upper corners after $t/d = 140$ lie outside the stated uncontaminated domain, and at $t/d = 150$ no open neighborhood of the origin satisfies the strict inequality. If a larger production domain was used, that value must be stated.

The uncontaminated part of every spacetime figure should be masked explicitly, and representative runs should be repeated at larger $L$ to demonstrate domain independence. Equation (29) implies an amplitude decay time $1/\operatorname{Im}\omega \simeq 1.4 \times 10^3 d$, almost an order of magnitude beyond the displayed collision time. This does not invalidate the pole, but it reinforces the need for longer, larger-domain evolutions before slowly leaking states, delayed escape, or persistent post-collision geometry can be classified.

## Minor and presentation comments

1. Equation (22) should be written separately at $z = -L$ and $z = +L$, with the corresponding integration constants and signs. As printed, a single expression containing only $-L$ is ambiguous. The text should also make clear whether equation (22) prescribes a Dirichlet value, supplies the value in the nonlinear Robin condition, or is only an asymptotic consistency check; imposing both a value and a normal derivative would overdetermine the wave equation.

2. The left asymptotic form in equation (10) omits an additive constant: direct expansion of equation (7) gives $A_{\mathrm{K}} = kx + A_{\mathrm{M}} + o(1)$ as $x \to -\infty$. Correspondingly, the antikink has $A_{\bar{\mathrm{K}}} = -kx + A_{\mathrm{M}} + o(1)$ on the right. The authors should either retain these constants or explain the separate time normalization used in each asymptotic region; the constants also enter the conformal-coordinate matching used for equation (22).

3. The dimensionless rescalings mentioned after equation (2) should be given explicitly. The units of $t$, $z$, $A$, $R$, $\phi$, and $\kappa$ should be unambiguous, as should any rescaling used on the curvature axes of Figures 5 and 6.

4. Section 3.2 should briefly define the perturbation variable and the self-adjoint domain underlying equations (23)–(26). In two-dimensional dilaton gravity there is no ordinary transverse graviton, so the later phrase “scalar and dilaton-metric perturbations” should be accompanied by a precise count of physical and constrained degrees of freedom.

5. Figure 3 would be much easier to assess if the authors overlaid the extracted window boundaries and $v_c$ on the heat maps. The three panels should use identical axis labeling and rendering, and the interpolation used to turn discrete velocity samples into a continuous-looking image should be stated.

6. Figures 5 and 6 need quantitative companion plots or tables. Line cuts of $e^A$ and $R$ at selected times, together with $\min e^A(t)$ and $\max |R(t)|$, would permit comparison across $\kappa$ without relying on perspective and color shading in three-dimensional surface plots.

7. The statement below equation (26) that the factorization implies non-negativity assumes the appropriate operator domain and vanishing boundary terms. A sentence stating those assumptions would complete the argument. The absence of positive embedded eigenvalues should likewise be justified or cited when concluding that no normalizable discrete mode exists.

8. The very long final paragraph on possible quantum extensions is premature relative to the unresolved classical mechanism. It should be shortened unless the revised paper demonstrates nonlinear excitation and decay of the pole in equation (29).

9. The institutional address contains visibly misencoded apostrophes. The authors should correct the character encoding in the publication version and perform a general typography check.

## Conclusion

The manuscript addresses a worthwhile nonlinear problem and contains a plausible new fully backreacting calculation. The field-equation algebra and the qualitative existence of an asymmetric leaking mode are convincing. The present evidence is not yet sufficient, however, to attribute the scattering trends to gravity, measure the claimed window evolution, or interpret the conformal-factor change invariantly. A revision that supplies controlled matter/gravity comparisons, constraint-consistent or demonstrably robust initial data, quantitative convergence and window tables, a reproducible resonance calculation, and a sharper literature discussion could make the work suitable for publication.