Referee Report

"Deriving the AdS$_3 \times S^3 \times T^4$ Quantum Spectral Curve I: Y-system and discontinuity relations"

Summary of the manuscript

The manuscript studies the relation between the mirror thermodynamic Bethe ansatz (TBA) and the conjectured quantum spectral curve (QSC) for the pure Ramond–Ramond AdS$_3 \times S^3 \times T^4$ superstring. Its stated aim is the first step of a two-paper program: to replace the nonlocal ground-state TBA by a local extended Y-system, consisting of functional Y-system equations together with analytic and monodromy data, and thereby prepare a derivation of the T-system and QSC in a companion work.

The starting point is a modified version of the ground-state TBA of Frolov and Sfondrini. The authors retain a single massless function Y_0, exchange the conventions for the two auxiliary functions $Y_\pm^{(\alpha)}$, and remove the factor $a(\gamma_{12}^{\circ\circ})$ displayed in equation (B.15) from the massless–massless dressing phase. Section 2 reviews the canonical and simplified equations and identifies a proposed $\mathbb{Z}_2$ "chiral" transformation exchanging the left and right massive wings, inverting and interchanging the auxiliary functions, and reversing the twists.

Section 3 gives two copies of the PSU$(1,1|2)$ Y-system in equations (3.1) and (3.2). The massless node does not belong to either T-hook; instead, it couples the two wings through the discontinuity data. The authors state the branch-point pattern, define long-cut discontinuities, and present monodromy equations for the auxiliary functions, Y_0, and the closest-cut discontinuities Δ_1 and $\bar{\Delta}_1$ of the massive functions. A particularly important claim is the local form (3.16), involving Y_0, the auxiliary functions, and a solution-dependent constant c.

Section 4 attempts the inverse construction. Cauchy representations reconstruct the auxiliary ratio and, after a lengthy separation into rational, Barnes, and BES contributions, the massless TBA equation. The auxiliary product is treated by reference to the analogous AdS$_5$ argument. The massive equations are then discussed in Section 4.4 through their analyticity strips, the simplified TBA, and the discontinuities (3.16), but the corresponding inversion is not written out. The appendices collect kinematics, scattering phases, kernels, kernel discontinuities, contour-residue calculations, the massless monodromies, and convolution identities used in the inversion.

The subject is important. Massless modes make AdS$_3$/CFT$_2$ integrability structurally different from its higher-dimensional counterparts, and a controlled bridge between the TBA and QSC would be valuable both conceptually and computationally. The separation into two T-hooks coupled only through monodromies, the explicit treatment of the massless cuts, and the detailed massless inversion are potentially significant results. However, several central formulae are presently inconsistent as printed, and the strongest equivalence and universality claims go beyond what is actually demonstrated.

Recommendation

I recommend **major revision**. I would be interested in a substantially corrected version, since the project is timely and the extended Y-system could become a useful foundation for the AdS$_3$ QSC.

Publication in JHEP or a comparable journal requires, at minimum, correction and propagation checks for equations (3.17)–(3.19) and (B.2), a complete or appropriately narrowed massive-particle inversion, a precise separation of ground-state results from conjectured excited-state statements, and a more rigorous account of the analytic hypotheses behind the contour deformations. The status of the altered massless dressing factor must also be established without using the desired Y-system analyticity as the sole justification.

Assessment of correctness and logical structure

The broad organization is sensible. Starting from canonical TBA equations, extracting first-sheet functional relations and monodromies, and then checking sufficiency by inversion is the correct strategy. The two-wing equations (3.1) and (3.2) have the expected $\mathrm{PSU}(1,1|2)^{\otimes 2}$ form, and it is physically natural that Y_0 enters through analytic continuation rather than as a node of either T-hook. The auxiliary inversion leading to equation (4.5) and the organization of the massless inversion into the five terms of equations (4.8) and (4.23)–(4.37) are useful and, at a structural level, persuasive.

The logical chain nevertheless fails at several important points. First, the two proposed expressions for c do not agree even on the twisted ground state from which they are derived. Second, the defining four-term combination for the improved BES phase is incorrect as printed. Third, the claimed reconstruction of all TBA equations is not performed for the massive sector: Section 4.4 gives a program and cites standard methods, whereas the conclusion describes a completed derivation. Finally, all discontinuities are derived from a twisted ground state, but some are promoted to arbitrary excited states by expectation, even though the logarithms then acquire additional zeros, poles, driving terms, and potentially unequal $\alpha = 1, 2$ data. These are not merely matters of exposition because they affect the asserted closure and state independence of the extended Y-system.

There are also numerous local inconsistencies in the massless appendices. The formula immediately before equation (G.29) is malformed, a continued rational kernel is equated to an undifferentiated scattering factor, and a residue sum over $1 \leq n \leq 2N$ is changed to $1 \leq n \leq N$ without an intervening identity. These errors occur precisely in the derivations supporting equations (3.10)–(3.14), so those results should be rechecked rather than corrected typographically in isolation.

Novelty and significance

The manuscript is well motivated by the unresolved relation between the pure-RR TBA and the QSC proposed by Cavaglià and collaborators and by Ekhammar and collaborators. The earlier TBA already supplied the canonical massive, massless, and auxiliary equations, while the recent QSC literature supplied the two $\mathrm{PSU}(1,1|2)$ Q-systems and highlighted the nonstandard massless analytic structure. The present paper's likely original contribution is the explicit extended Y-system joining these descriptions: in particular, the auxiliary and massless discontinuities, the proposed local form of Δ_1 and $\bar{\Delta}_1$, the chiral rewriting, and the explicit inversion of the massless equation.

Those results would be of clear interest if made internally consistent. They could clarify how degrees of freedom that are not assigned to T-hook nodes are nevertheless encoded in QSC monodromies, and they may provide the right starting data for efficient finite-coupling calculations. The current presentation should, however, distinguish three levels of novelty and certainty more sharply: results algebraically derived from the modified ground-state TBA; conjectured promotion of those results to excited states and to the singular $\mu \to 0$ limit; and consequences deferred to the unpublished companion paper. A short comparison table with the inputs taken from the earlier TBA, the ingredients already conjectured in the QSC literature, and the genuinely new equations proved here would make the advance easier to assess.

Major comments

1. **The normalization of the constant c is internally inconsistent.**

   Equation (3.17) defines

   $$c = \frac{1}{4}\log\frac{1+Y_m}{1+\dot{Y}_m} \star \big(K_m(2-v) + K_m(2+v)\big).$$

   The next equation states, for either auxiliary species,

   $$\log\big(Y_-^{(\alpha)}Y_+^{(\alpha)}\big)(\pm 2) = -\log\frac{1+Y_m}{1+\dot{Y}_m} \star K_m(v \mp 2).$$

   Since the two auxiliary species obey identical ground-state equations, inserting this relation into equation (3.19) gives

   $$-\frac{1}{2}\sum_{\alpha=1}^{2}\Big[\log\big(Y_-^{(\alpha)}Y_+^{(\alpha)}\big)(2) + \log\big(Y_-^{(\alpha)}Y_+^{(\alpha)}\big)(-2)\Big] = 4c,$$

   assuming the evenness of K_m used elsewhere in the manuscript. Thus equation (3.19), as written, does not reproduce equation (3.17); it differs by a factor of four on the very solution used in the derivation.

   This discrepancy directly affects the local discontinuities (3.16), their symmetrized versions (3.20)–(3.25), the chiral transformation $c \mapsto -c$, and the massive inversion. With the displayed equations, the ground-state endpoint formula would be $-\frac{1}{4}$ times the bracket for one auxiliary species, or equivalently $-\frac{1}{8}$ times the sum over the two identical species. The authors should fix the coefficient, explain whether the endpoint expression is a sum or an average over α, and derive the excited-state generalization rather than selecting it by expectation. All later formulae containing c should then be rechecked. A simple numerical evaluation on a twisted ground-state solution would be a useful independent normalization test.

2. **The improved BES phase is defined incorrectly in equation (B.2).**

   The four-term expression in equation (B.2) contains the term

   $$-\widetilde{\Phi}(x_1^{+n}, x_2^{-n'})$$

   twice. The complementary mixed term involving x_1^{-n} and $x_2^{+n'}$ is absent. The standard fused phase has the alternating combination

   $$\widetilde{\Phi}(x_1^{+n}, x_2^{+n'}) + \widetilde{\Phi}(x_1^{-n}, x_2^{-n'}) - \widetilde{\Phi}(x_1^{+n}, x_2^{-n'}) - \widetilde{\Phi}(x_1^{-n}, x_2^{+n'}).$$

   The printed expression does not have the required antisymmetric/fusion structure and cannot support the claimed equivalence to the cited BES representation.

The expanded BES kernel in Appendix E appears to contain the correct four distinct denominators, suggesting that later algebra may have used the intended expression. Nevertheless, because BES kernels are subsequently used in Sections 4.3, E, and G, this cannot be corrected silently. The authors should fix (B.2), state which expression was actually used in every kernel calculation, and explicitly verify the double-convolution identities and discontinuities that feed equations (4.8), (4.36), and (4.37). Checks of unitarity, crossing, and the $n \to 0^+$ limit should be included.

3. **The manuscript does not actually reconstruct all massive TBA equations.**

   The abstract, introduction, and conclusion claim full equivalence between the extended Y-system and the ground-state TBA. Section 4.4 instead says that the reconstruction "can be achieved," summarizes standard steps, and concludes that one "should apply" the canonical-kernel method. It does not derive the massive simplified equations with their driving terms and asymptotics, does not insert (3.16) to obtain the displayed equations (2.17), and does not carry the simplified equations back to the canonical equations (2.7) and (2.8). The inversion of $\log(Y_- Y_+)$ in Section 4.1.2 is similarly referred to the AdS$_5$ calculation without showing how the two wings and the present cut conventions alter it.

   This gap is central because the local formula for Δ_1 is one of the paper's most novel claims and because the normalization problem above enters precisely here. The authors should supply the complete massive inversion, including the large-u data that fix homogeneous solutions and all contour residues, and demonstrate explicitly that the canonical kernels and twist factors are recovered. If the calculation is to be deferred, then the main claim must be narrowed to equivalence for the auxiliary and massless subsectors plus a proposed completion for the massive sector. The current conclusion that "all TBA equations" have been reconstructed is not supported by Section 4.

4. **Ground-state derivations are being promoted to a universal excited-state system without the required source terms.**

   The discontinuities are derived from a twisted ground-state TBA, for which the two auxiliary species satisfy identical integral equations and for which zeros and poles may be kept away from the chosen deformation contours. The paper then assumes the same branch pattern for excited states, promotes equation (3.19) to a universal definition, and describes the resulting system as state independent. This promotion is not established. Excited-state TBA equations generally contain contour-deformation residues from Bethe roots, and $\log(1+Y)$, $\log Y$, and the auxiliary logarithms acquire extra cuts or additive driving terms. These can change both the Cauchy inversions and the endpoint constants.

   The unresolved α dependence makes the problem visible already in the printed equations. Equation (3.12) combines an unsummed expression containing $Y_{\pm}^{(\alpha)}$ with an explicit sum over α, while Appendix G obtains factors such as $2\log(Y_-^{(\alpha)}/Y_+^{(\alpha)})$. Such formulae are unambiguous only when the two species are equal, yet the footnote below equation (3.18) explicitly anticipates excited states for which they are unequal. Likewise, the $\mu \to 0$ protected vacuum is acknowledged to be a singular solution, but no limiting argument is given.

   The authors should choose one of two paths. They may rigorously restrict this paper to the twisted ground state, removing universal and all-state claims. Alternatively, they should derive the excited-state source terms, state the contour prescription around Bethe roots, replace every free α by a definite sum or species-specific relation, and prove that the local monodromy equations survive. Analogy with AdS$_5$ is useful motivation but is not a proof in a model whose novelty is precisely its massless analytic structure.

5. **The analytic hypotheses behind the Cauchy inversions are incomplete and are not checked for the functions used.**

   Equations (4.2), (4.15), and the subsequent contour deformations apply Cauchy's theorem to logarithms divided by a square root. Besides the horizontal branch cuts listed in Table 1 and the vertical cut of $\log Y_0$, the logarithms can have zeros, poles, and branch points associated with $1 + Y_i = 0$ and with the twisted auxiliary factors. Such singularities give residue or source terms. The statement near equation (4.27) that the kernel has no enclosed pole does not establish that the full logarithmic integrand is analytic inside the contour.

   The suppression of the large circle is stated for the auxiliary ratio but not demonstrated for $\log Y_0$, and the exchanges of infinite sums, contour shifts, and integrations require convergence estimates. Properties 1 and 2 in Appendix H list strip analyticity but do not state decay or summability conditions, and the first-strip cancellation used in the proof of Property 1 is not covered if the integer n in its second hypothesis really begins at $n = 1$.

   The revision should formulate a precise inversion lemma including allowed zeros/poles, large-u asymptotics, logarithm branches, and convergence hypotheses, then verify it for each ground-state function. If the relevant properties follow from the TBA, they should be stated and proved or supported by a controlled solution. The same analysis should make clear which new residues appear for excited states. Without these data, the Cauchy representation determines a function only up to omitted source and homogeneous contributions.

6. **The removal of the massless factor (B.15) requires a non-circular physical justification.**

   The modified massless–massless phase is an input to every later result. The manuscript argues that the factor $a(\gamma_{12}^{\circ\circ})$ used in the earlier TBA produces nonstandard cuts, is incompatible with the proposed QSC, and has no known mixed-flux extension. This is suggestive, and the cited work on exchange relations may indeed resolve the crossing sign. However, deleting a factor from an S-matrix because it obstructs the desired analytic Y-system risks making the subsequent agreement circular.

   The authors should demonstrate explicitly that the phase used here satisfies the appropriate unitarity and crossing equations with the claimed exchange relations, identify whether the change is a CDD transformation or a convention change, and explain why finite-volume energies are unaltered or how they change. The derivation should also show exactly which extra branch points (B.15) would generate and why they are physically inadmissible, rather than only unlike standard AdS/CFT Y-systems. Since the paper starts from a TBA different from the one it cites, the status of this modification is part of the principal result, not a peripheral convention.

7. **The massless discontinuity appendix contains errors and unexplained changes that affect the main monodromy equations.**

   Several points in Appendix G require a fresh derivation.

   - The equation immediately before (G.29), which is the advertised final real-axis result, has an unmatched parenthesis after $K_{-}^{ny}(v, u)$; as typeset, the dotted-wing term becomes part of the second argument of that kernel. It therefore does not equal twice the auxiliary TBA equation as claimed.

- After defining K^{n0}_{Rational} as a logarithmic derivative, the continuation below the cut is written as an undifferentiated factor proportional to $i\sqrt{x^-/x^+}\,(x^+ - x)/(xx^- - 1)$. A kernel cannot equal the corresponding S-matrix factor. The continued object and its derivative must be distinguished.
- In the upper- and lower-half-plane calculation, the text says that residues occur for every $n = 1, \ldots, 2N$, but the very next discontinuity formula contains a sum only to N. No fusion, parity, or cancellation identity is supplied to account for the missing terms.
- The BES contribution and the final formulae again use a single free auxiliary index multiplied by two, although the main text later seeks a species-dependent excited-state formulation.

These issues feed directly into equations (3.10)–(3.14) and then into the inversion (4.10)–(4.37). The authors should correct the printed equations, show the residue bookkeeping step by step for general N, and test at least the $N = 1$ and $N = 2$ relations by direct analytic continuation of the original TBA kernels. The corrected monodromies should then be propagated through Section 4.

8. **The chiral symmetry requires control of logarithm branches and additive constants.**

   The multiplicative rearrangement used around equation (2.23) is algebraically valid after the product over α cancels phases and signs, but its logarithm is defined only modulo $2\pi i$. Similar branch choices enter the inversion of $Y_\pm^{(\alpha)}$, the transformation of c, and the twisted factors in equations (2.7)–(2.9). The paper treats the logarithmic equalities as exact without specifying a continuous branch or showing that possible constants vanish after convolution.

   Because this $\mathbb{Z}_2$ symmetry is later used as a consistency condition on the extended Y-system and QSC, the authors should prove it at the level of the exponentiated TBA equations or specify the logarithm branches and asymptotic normalization that remove the ambiguity. The proof should include the massless equation and the endpoint definition of c after its normalization is corrected. A direct check on a twisted numerical solution would again be valuable.

Minor and presentation comments

1. Equation (4.12) is described as the TBA expression for $-\sum_\alpha \log(Y_-^{(\alpha)}/Y_+^{(\alpha)})$, but comparison with equation (2.11) shows that the displayed combination instead corresponds to the summed ratio together with the additional local $2\log(1+Y_0)$ term used in equation (3.12). The prose should be corrected and the signs checked.

2. The figure caption in Section 3.1 refers to the chiral symmetry with an equation-style reference to a section label. It should read “Section 2.4” rather than display the section number as an equation.

3. The ground-state energy formula following equation (2.11) should display the sum over massive bound-state number and use p_n rather than an unindexed p. As written, the repeated n is only implicit.

4. The domain statement in Appendix H that $u \in (-2, 2)$ “with $\operatorname{Im} u > 0$” is contradictory if the interval is meant literally. The intended condition is presumably $-2 < \operatorname{Re} u < 2$ with a specified positive imaginary regulator.

5. In the pole analysis leading to equation (H.39), the sentence concerning $t + in/h$ refers a second time to $\gamma^{-n}(t)$; the second occurrence should presumably be $\gamma^{+n}(t)$.

6. The unitarity relation in Appendix B writes $S^{n0}(u_2, x_1)$ in one factor, mixing a rapidity and a Zhukovsky variable in an argument slot otherwise occupied by rapidities. This should be corrected or explained.

7. The statement following equation (F.7) says that the bracketed expression is the TBA equation for $\log Y_0$ but cites the right-massive equation. It should cite the massless equation (2.9).

8. Equation (F.21) contains an unsummed α in the general dotted-wing discontinuity and ellipses for residues, whereas the specialized $n' = 1$ formula contains a product over both species. The relation between the two expressions should be made explicit.

9. The source produces duplicate equation destinations around equations (2.33) and (4.30), and several displayed equations are built with deprecated or fragile alignment constructs. The authors should compile with warnings visible, remove duplicate anchors, and replace the affected `eqnarray`-style displays with `align`.

10. Section 2 alternates between $\dot{Y}_n$ and $\bar{Y}_n$ notation, while later appendices use both dotted and barred symbols. A single notation should be enforced, especially in equations involving ratios of the two wings.

11. The phrase "purely imaginary chemical potential" in the conclusion should be reconciled with the convention $e^{i\mu_\alpha}$ and the earlier description of μ as a twist parameter. State whether μ is real and the thermodynamic chemical potential is imaginary, or whether μ itself is analytically continued.

12. The analytic character of the branch points is described as an expectation in Section 3.2. Table 1 should distinguish established locations from assumed logarithmic order, and the inversion should state which conclusions depend on that order.

13. The paper relies repeatedly on an in-preparation companion work. This is acceptable for future applications, but every claim of the present paper must be independently checkable. In particular, the role of c and the regularity conditions needed here should not be deferred to the companion.

14. A notation table for the four convolution symbols, the two analytic-continuation paths, the several massive/massless kernels, and the dotted/undotted wings would materially improve readability. The present 60-page derivation is difficult to audit because locally similar kernels differ by hats, tildes, and argument-sheet prescriptions.

Conclusion

The manuscript addresses an important missing step in AdS$_3$/CFT$_2$ integrability and contains a substantial amount of potentially useful technical work. The two-T-hook organization and the attempt to reconstruct the massless TBA from local monodromies are promising. At present, however, the internal factor-of-four discrepancy in c, the incorrect BES definition, the unperformed massive inversion, and the unresolved ground-state/excited-state distinction prevent the central correctness and equivalence claims from being accepted. A revision that corrects these formulae, rechecks the affected discontinuities and kernel identities, supplies the missing massive derivation, and states precise analytic hypotheses could make the work suitable for publication.