Referee Report on “Hodge Numbers for Orbifolds of Calabi–Yau Threefolds Ferma Type and Roan’s Pairs”

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Summary and overall assessment

The manuscript studies diagonal orbifolds of Fermat-type Calabi–Yau threefold hypersurfaces. Starting from exponent tuples

$$\sum_{i=1}^{5} \frac{1}{a_i} = 1,$$

it describes the diagonal symmetry group $G^{\max}$, the grading subgroup G_0, and Krawitz-dual admissible groups G and G^T. Polynomial deformations invariant under G are encoded by degree-one elements of the Jacobian ring. The proposed new ingredient is a count of “Roan pairs”: ordered, support-orthogonal pairs of level-one elements with respectively two and three zero coordinates. Definition 5.1 adds these pair counts to the invariant-deformation counts and calls the results Roan Hodge numbers. Theorem 5.2 claims, for all 147 Fermat-type Calabi–Yau threefold weight systems and all admissible diagonal quotients, that their difference reproduces one half of the Vafa Euler number. Section 6 illustrates the rule for four quintic quotients. Theorem 7.3 and Section 8 then identify ten Fermat quotients whose proposed Hodge pairs equal those of selected Borcea–Voisin threefolds. The conclusion proposes a further monotonicity conjecture for Euler numbers.

An explicit combinatorial realization of twisted sectors, together with a complete and reusable database, could be useful. The elementary quintic examples are informative, and several representative counts are internally consistent. The attempt to connect the combinatorics with heterotic massless states could also be interesting if developed into an actual state-space correspondence.

The present manuscript does not, however, establish its main claims to journal standard. Theorem 5.2 has no proof beyond the sentence that it was checked numerically, while the cited database [DB] has no URL, archival identifier, tables, code, or even a precise count of the cases tested. More fundamentally, agreement of an Euler-number difference does not prove that the two separately defined counts are stringy or orbifold Hodge numbers. No age-graded fixed-sector construction or comparison with Chen–Ruan/FJRW cohomology is given. Several case lists in Section 8 also contradict Definition 5.1 as printed: in Sections 8.3 and 8.4 the two orientations of the Roan-pair counts are reversed, and multiple further tuples and pair labels are corrupted by apparent copy-and-paste errors. Finally, Theorem 7.3 establishes only equality of two integers, while parts of Section 8 infer mirror or birational relations that do not follow from equality of Hodge numbers.

Recommendation: reject in the present form, with resubmission encouraged after a fundamental revision. A viable resubmission would need a rigorous identification of the proposed counts with standard stringy Hodge sectors (or substantially narrower terminology), a reproducible and independently checkable proof of the exhaustive finite claim, corrected machine-generated case data, and a much more accurate account of novelty relative to existing Landau–Ginzburg orbifold and Borcea–Voisin/BHCR results.

Correctness and logical structure

The broad combinatorial strategy is plausible. For a Fermat potential, the degree-one part of the Jacobian ring is naturally represented by exponent vectors $0 \leq S_i \leq a_i - 2$, and Krawitz orthogonality provides a convenient test of invariance. The quintic calculations in Section 6 illustrate how a mismatch between invariant-deformation counts can be repaired by support-disjoint level-one pairs. For example, the second quintic quotient has deformation counts 25 and 5, together with 24 pairs in one orientation and none in the other, giving the stated Euler number -88. This is a useful consistency check of the proposed rule.

The logical gap is that these checks validate only a numerical difference. Equations (5.1) and (5.2) simultaneously define augmented counts and impose their mirror interpretation. Theorem 5.2 then checks

$$\frac{\chi}{2} = h^{2,1}(\text{mirror}) - h^{2,1}(\text{original}),$$

which is a single scalar identity. It cannot establish that either summand is the corresponding $h^{2,1}$, nor that its mirror partner is $h^{1,1}$. A proof at the level of fixed loci, ages, and bidegrees is therefore essential.

There is also a serious distinction among the affine Landau–Ginzburg potential, the projective weighted hypersurface, the quotient stack, the coarse geometric quotient, and a crepant resolution. The manuscript uses the same notation W/G for several of these objects. Since G_0 is the exponential grading group and acts trivially on the projective hypersurface, the effective geometric quotient is normally by G/G_0, although the Landau–Ginzburg orbifold is labelled by G. Until this distinction is made, the interpretation of equations (1.1), (3.1), (4.1), and (4.2) as Euler numbers of $\widetilde{W/G}$ is not precise.

Novelty and significance

The manuscript acknowledges that Roan proved the relevant combinatorial identity when all $a_i \geq 3$, and it cites Krawitz for Landau–Ginzburg mirror state spaces. The potentially new contribution is thus not the Vafa formula, BHK Hodge mirror symmetry, or orbifold Hodge theory itself. It is the claimed extension of a particularly explicit Roan-pair decomposition to the Fermat weight systems containing an exponent 2, together with an exhaustive database and ten numerical Borcea–Voisin matches.

That contribution could be worthwhile, but its novelty is not presently isolated. The paper should state how many of the 147 exponent systems are outside Roan's hypothesis, how many inequivalent admissible groups occur, what equivalence relation is used, and what new phenomenon makes the $a_i = 2$ cases nontrivial. It should compare the resulting spectra with the earlier abelian $c = 9$ Landau–Ginzburg orbifold classifications and algorithms, including the Kreuzer–Schimmrigk–Skarke and Kreuzer–Skarke work. It should also explain what the Roan-pair decomposition adds beyond the known Hodge pairs.

The relevant standard result is a full bigraded state-space statement, not only Euler-sign reversal. Krawitz's state-space mirror construction and the Chiodo–Ruan Landau–Ginzburg/Calabi–Yau state-space isomorphism already relate the BHK model to age-graded orbifold cohomology. The manuscript would make a genuine conceptual advance if it proved that Roan pairs give an explicit basis or canonical parametrization of the corresponding twisted sectors. Without that theorem, "Roan Hodge numbers" are at most candidate combinatorial counts.

The Borcea–Voisin claim is similarly weaker than the existing literature. The cited work of Artebani, Boissière, and Sarti establishes projective birational Borcea–Voisin models compatible with BHCR transposition in the relevant setting. Theorem 7.3 only finds Fermat quotients with the same ordered Hodge pair. Matching two integers may be useful catalogue information, but it is not by itself a geometric relation, a mirror construction, or a birational correspondence.

Major comments

1. **Theorem 5.2 requires an actual proof or a fully reproducible exhaustive certificate.**

   The complete proof of the central theorem is the statement that the formula was verified by numerical computations, with details placed in [DB]. The bibliography entry [DB] gives only the authors and a title. It has no URL, DOI, arXiv number, repository commit, version, or machine-readable supplement. The manuscript does not state the number of admissible subgroups considered, how groups related by coordinate permutations were identified, how Krawitz duals were generated, whether exact rational arithmetic was used, or how the enumeration was independently checked. The Introduction alternates between "proved," "checked," and "computer verification," which are not interchangeable claims.

   A finite exhaustive computation can constitute a proof, but the authors must specify the finite search space, prove that the enumeration covers it, describe the algorithm unambiguously, deposit executable code and complete output under a stable identifier, and provide compact certificates or independent cross-checks. The main paper should include summary tables by exponent type and group order rather than asking the reader to trust hand-pasted lists. The theorem also cites equation (4.1) for the Vafa number, although (4.1) is the G_0 specialization; the formula for arbitrary G is equation (4.2). This is not a harmless cross-reference: for the second quintic quotient the right-hand side of Theorem 5.2 is $5 - 49 = -44$, while equation (4.1) gives $-200/2 = -100$; equation (4.2) gives the intended $-88/2 = -44$.

2. **Definition 5.1 does not establish stringy Hodge numbers.**

   The opening of Section 5.1 identifies $|S^G|$ with $h^{2,1}(W/G)$, even though W/G is described as singular and the meaning of this Hodge number has not been fixed. Equations (5.1) and (5.2) then add pair counts and set the results equal to mirror $h^{1,1}$ values by definition. Theorem 5.2 checks only their Euler difference. Equality of the difference with the Vafa invariant cannot determine either Hodge number separately; many pairs of integers have the same difference.

   The required result is an identification theorem. For every contributing pair, the authors should identify the corresponding fixed locus or Landau–Ginzburg sector, compute its age and left/right charges, show that it contributes with multiplicity one in the asserted bidegree, and prove that all twisted-sector classes arise exactly once. This should yield a comparison with Chen–Ruan/stringy Hodge numbers or the FJRW/Krawitz bigrading. If the authors cannot provide such a comparison, the quantities should be renamed "Roan pair counts" or "candidate Hodge counts," and all claims that they are the Hodge numbers of a resolution should be removed.

3. **The quotient and resolution whose invariant is being computed must be defined precisely.**

   Section 2 moves without warning among a polynomial W, its affine zero locus, and a projective hypersurface. The assertion in Section 2.1 that the quotient has a "unique singularity at zero" refers, if anything, to the affine cone: the origin is not a point of the weighted projective hypersurface. Similarly, G_0 acts trivially after projectivization, so a geometric quotient should involve the effective group G/G_0, whereas G is the natural label of the Landau–Ginzburg orbifold.

   The authors should separately define the affine potential, the quasismooth projective Calabi–Yau hypersurface, the quotient stack, and its coarse space. They should state whether χ in equations (1.1), (3.1), and (4.2) is Vafa's Landau–Ginzburg orbifold Euler number, a stringy Euler number of the coarse quotient, or the ordinary Euler number of a specified crepant resolution. The hypotheses guaranteeing equality of these quantities, existence of a crepant resolution, and independence of the chosen resolution must be cited or proved. The notation $\widetilde{W/G}$ is otherwise too ambiguous to support Theorem 5.2.

4. **The Roan-pair rule needs a derivation, not only a counting prescription.**

   Definition 5.1 uses canonical nonnegative representatives and requires

   $$\sum_i z_i' a_i z_i = 0.$$

   Because the representatives are nonnegative, this is exactly a disjoint-support condition; it is much stronger than the integrality condition defining Krawitz duality in equation (2.4). The manuscript does not explain why precisely the 2-zero/3-zero split, exact zero pairing, and ordered-pair multiplicity encode all and only the relevant twisted classes. It also does not address possible stabilizers or repeated descriptions of the same sector.

   The revision should derive the rule from a standard sector formula, state the representative convention and ordering explicitly, and prove injectivity and surjectivity of the proposed parametrization. The unnumbered lemma in Section 5.2 is a simple useful observation, but "the proof is obvious" should be replaced by the calculation

   $$\sum_i Z_i = \sum_i z_i - \sum_i \frac{1}{a_i} + \#\{i : z_i = 0\} = \#\{i : z_i = 0\},$$

   using equation (2.1) and the level-one condition. This lemma alone does not supply the missing cohomological identification.

5. **Sections 8.3 and 8.4 contradict Definition 5.1 as printed.**

   The problem can be seen without any geometric interpretation. In Section 8.3, with $G = G^{\max}$ and $G^T = G_0$, the manuscript gives 17 G-invariant and 29 G^T-invariant deformations. It then labels the two pair counts as 6 for (G^T, G) and 18 for (G, G^T). Applying equations (5.1) and (5.2) literally gives

   $$h^{2,1}(\widetilde{W/G}) = 17 + 6 = 23, \qquad h^{1,1}(\widetilde{W/G}) = 29 + 18 = 47,$$

   not the claimed $(35, 35)$. Interchanging the two orientation labels repairs the arithmetic: $17 + 18 = 29 + 6 = 35$.

   Section 8.4 has the same failure. The printed counts $13, 21, 6, 14$, with the printed orientation labels, give $(h^{2,1}, h^{1,1}) = (19, 35)$, whereas swapping the pair orientations gives the advertised $(27, 27)$. Thus the numerical lists appear to support the target values only after correcting the labels. Since these case lists are offered as the proof of Theorem 7.3 and as evidence for Theorem 5.2, the error is substantive. Every example should be regenerated directly from the deposited code, with column headings tied unambiguously to equations (5.1) and (5.2).

6. **The ten-case data contain further material inconsistencies.**

   There are enough independent transcription and group-order errors to undermine confidence in the claimed classification. In the list preceding Theorem 7.3, the fourth tuple is printed as $(3, 4, 4, 10, 20)$, which does not satisfy equation (2.1); Section 8.4 uses the Calabi–Yau tuple $(3, 4, 4, 10, 15)$. The same subsection alternates the orders $(4, 4, 3, 10, 15)$ and $(3, 4, 4, 10, 15)$. For $(3, 4, 4, 7, 42)$, Section 7 names $\mathbb{Z}_{42} \times \mathbb{Z}_2$, but $d = 84$ and the displayed maximal group in Section 8.3 has order 168, namely $\mathbb{Z}_{84} \times \mathbb{Z}_2$. For $(4, 4, 4, 6, 12)$, Sections 7 and 8.6 claim $\mathbb{Z}_{24} \times \mathbb{Z}_2$, while $d = 12$ and the displayed generators produce a group of order 24, consistent with $\mathbb{Z}_{12} \times \mathbb{Z}_2$.

   Section 8.6 also labels both its 3-pair and 10-pair lists as (G, G^T), although the target Hodge pair requires opposite orientations, and its concluding formula names $(4, 4, 4, 5, 20)$ rather than that subsection's $(4, 4, 4, 6, 12)$. Section 8.7 ends with $W_{3,4,4,5,8}$ instead of $W_{3,4,4,8,24}$. Section 8.9 labels both the 6- and 18-pair lists in the same orientation and ends with $W_{3,4,4,12,12}$, although the case is $(4, 4, 5, 5, 10)$;

its displayed decomposition $15 = 5 + 10$ also does not match the listed 9 deformations and 6 pairs. Section 8.5 calls a chain-type BV mirror polynomial the K3 surface itself, contrary to the Fermat K3 identified in Section 7, while Section 8.10 calls $(4, 4, 3, 9, 18)$ a K3 type rather than distinguishing the K3 exponents $(2, 3, 9, 18)$ from the associated threefold $(3, 4, 4, 9, 18)$. A pair with denominators belonging to $(3, 4, 4, 7, 42)$ is stranded at the end of the Section 8.2 list.

These should not be repaired one at a time by hand. The revision should generate a compact table containing, for every case, G, G^T, the two deformation counts, the two oriented pair counts, the two augmented Hodge counts, and the independently evaluated Vafa number. The displayed examples can then be selected automatically from the same data.

7. **Theorem 7.3 proves only a numerical coincidence, and later geometric conclusions do not follow.**

   The statement of Theorem 7.3 says that for each of ten Fermat types there exists a group whose two proposed Hodge numbers coincide with those from Proposition 7.2. Even if all counts are corrected, this proves only equality of an ordered pair of integers. It does not identify the Fermat quotient with the Borcea–Voisin threefold, construct a mirror map, or imply birationality.

   Section 8.1 nevertheless says that two proposed mirrors "are birational" after observing the same Hodge data. Equal Hodge numbers do not imply birationality. The "multiple mirror" conclusion in Section 8.3 likewise needs a geometric statement, not merely different decompositions of equal integers. The authors should either construct explicit twist maps, birational models, or stack/LG correspondences and compare them with the results of Artebani–Boissière–Sarti, or consistently weaken the claim to ten numerical Hodge-pair matches. The latter would still be legitimate catalogue information, but it should not be advertised as a new relation between the two mirror constructions.

   The geometric review in Section 7 should also be corrected. Blowing up $S \times E$ along the fixed curves does not itself produce a Calabi–Yau threefold: for a codimension-two blow-up the canonical class acquires the exceptional divisor. It is the subsequent quotient/crepant-resolution construction that yields the Borcea–Voisin Calabi–Yau. The exposition around Theorem 7.1 should track these operations and canonical divisors accurately.

8. **The literature comparison is insufficient to support the novelty claim.**

   The statement in the Introduction that Krawitz "invented orbifold Hodge numbers" is historically and mathematically inaccurate. Stringy and orbifold Hodge theories predate that work, and Krawitz's contribution is a specific bigraded Landau–Ginzburg mirror state-space construction. Directly relevant comparisons include Batyrev–Dais, Chen–Ruan, Poddar's orbifold Hodge calculation for Calabi–Yau hypersurfaces, Borisov–Libgober, Krawitz, and Chiodo–Ruan. Early abelian Landau–Ginzburg orbifold spectrum catalogues are especially relevant to the claimed exhaustive database.

   The manuscript should also explain precisely what extends Roan's theorem: identify the cases with $a_i = 2$, state why his argument does not apply, and show what new argument or data resolves them. Finally, [ABS] is described as merely "another approach," although it proves a stronger birational compatibility between Borcea–Voisin and BHCR constructions. The revised paper needs a case-by-case comparison with that result and a clear statement of what remains new.

9. **The claimed physical interpretation remains motivational rather than demonstrated.**

   Remark 5.4 says that Roan pairs are realized by massless heterotic vertex operators, and Remark 6.1 identifies the data with Batyrev lattices and Borisov cohomology. Neither assertion is accompanied by a map, a charge/age computation, or a state-by-state example. For a high-energy theory audience, this is the principal possible payoff beyond a database of algebraic-geometric counts.

If this interpretation is part of the paper's contribution, the authors should work through at least one nontrivial quotient: map an invariant deformation and a Roan pair to the corresponding **27** or $\overline{\mathbf{27}}$ massless sector, match left/right $U(1)$ charges and multiplicities, and explain how the OPE or BRST/cohomological construction distinguishes the two orientations. Otherwise the physics claims should be reduced and the work presented candidly as a computational classification paper.

10. **The conclusion's monotonicity conjecture needs a precise evidentiary basis.**

    The unnumbered conjecture is stated in arbitrary dimension for nested groups $G \subset G'$, subject to $|G'| \leq \sqrt{\prod_i a_i}$, but the evidence cited comes from the undisclosed threefold database. The origin of the order bound is not explained, nor is it clear whether $|G|$ means the Landau–Ginzburg group or the effective projective group. The preceding claims about log-convexity and log-concavity list triples of numbers without identifying the nested group chains or their ordering.

    The authors should state the exact domain in which the conjecture was tested, explain the self-dual order threshold, list the relevant group chains for the counterexamples to log-convexity/concavity, and avoid extrapolating to arbitrary m without supporting structure. This conjecture is secondary to Theorem 5.2, but in a corrected paper it could be an interesting, clearly delimited observation.

Minor comments

1. The standard term is "Fermat type," not "Ferma type." The title and manuscript use "Ferma" throughout. There are also pervasive misspellings of Berglund–Hübsch–Krawitz, "orbifold," "threefold," and related terminology. A complete technical copyedit is required after the mathematical revision.

2. The setup following equation (2.1) prints $k_i = ta_m/a_j$, where the denominator should be a_i. The subsequent Gorenstein discussion says $d/k_m = t$, but $k_m = t$ and $d = ta_m$ give $d/k_m = a_m$. The range $j = 1, \dots, a_{m-1}$ is also nonsensical and appears to mean $j = 1, \dots, m$.

3. Section 2.2 calls $\prod_i \mathbb{Z}_{a_i}$ the full automorphism group. It is the full diagonal symmetry group of the Fermat polynomial; equal exponents also permit coordinate permutations, and "full automorphism group" is a stronger statement.

4. Equation (2.4) mixes integer lifts u_i, v_i with symbols g_i, g_i' that have not been defined as rational coordinates. State a canonical lift or prove independence of the chosen lifts. The Krawitz dual should be defined as the full annihilator under this pairing, not merely by a one-way pairwise orthogonality condition.

5. Equation (2.5) is malformed and appears to declare the image of all deformations to be a subset of an arbitrary G^T. The image is naturally a subset of $G^{\max}$; Definition 2.1 should then intersect it with G^T. This correction is important for the invariant-deformation logic.

6. The displayed formula in Section 2.4 for the phase of $g_0^b(x^S)$ is not the coordinatewise reduction it claims to be. Without reducing representatives the exponent is simply $b\sum_i S_i/a_i = b$; with coordinatewise reduction the integer correction involves $\sum_i \lfloor b/a_i \rfloor S_i$, not only the indices for which b/a_i is integral. The invariance conclusion is true, but the derivation should be corrected.

7. Remark 2.2 mixes exponent and phase coordinates. With

$$B = \{z_i \geq -1 : \sum_i k_i z_i = 0\},$$

the shift to nonnegative exponent coordinates is by the all-ones vector, not by $(1/a_1, \ldots, 1/a_m)$. The coordinate convention should be made uniform.

8. In equation (1.1), equation (3.1), and equation (4.2), replace the prose condition on the product by a precise set such as
$$\{i : (g_1)_i = (g_2)_i = 1\}.$$
The current wording can be read as equating g_2, rather than its i-th coordinate, to 1.

9. Theorem 7.1 and Proposition 7.2 should state the hypotheses and exceptional fixed-locus cases in Nikulin's classification. The triplet (r, a, δ) determines the relevant invariant-lattice/deformation type, not the isomorphism class of an individual K3 surface as asserted in Section 7. In particular, Proposition 7.2 as printed misses the fixed-point-free Enriques case $(r, a, \delta) = (10, 10, 0)$: Theorem 7.1 gives $(h^{1,1}, h^{2,1}) = (11, 11)$, whereas the displayed formula in Proposition 7.2 would give $(15, 15)$. The ten applications may avoid this case, but the proposition needs the exception.

10. The second item in the ten-type list repeats the same monomial $z_3 z_5^6$ twice in the proposed mirror polynomial. All displayed potentials should be checked for weighted homogeneity and nondegeneracy.

11. The four quintic cases in Section 6 and ten cases in Section 8 would be far clearer as tables. Empty subsection titles, raw Python-style arrays, and very long unpunctuated lists make it difficult to see the logical content and contributed to the orientation errors.

12. The bibliography entry [Roan] appears to mistitle the work as "The Minor of Calabi–Yau Orbifold" rather than "The Mirror of Calabi–Yau Orbifold." Entry [DB] is not a usable reference. Several entries need complete journal data, stable hyperlinks, and corrected character encoding.

13. The Introduction's heterotic-spectrum summary should call both the $\mathbf{27}$ and $\overline{\mathbf{27}}$ matter multiplets chiral supermultiplets; describing the Kähler-modulus contribution itself as "vector-like supermultiplets" is misleading. The net chirality, rather than either separate Hodge number, is related to $\chi/2$.

Recommendation

The manuscript contains the seed of a useful specialized result: an explicit combinatorial parametrization of twisted contributions for Fermat diagonal orbifolds, potentially accompanied by a comprehensive database. In its current form, however, the central theorem is neither proved nor reproducible, the proposed counts are not identified with genuine stringy Hodge numbers, and the printed case data contain contradictions that directly affect the advertised results. The Borcea–Voisin discussion also claims more geometry than equality of Hodge pairs can establish.

I therefore recommend rejection in the present form. I would view a substantially reconstructed resubmission positively if it (i) proves a bigraded sector correspondence or adopts narrower terminology, (ii) supplies archived code, data, and exact certificates for the exhaustive computation, (iii) regenerates and corrects all examples, (iv) accurately positions the result relative to Roan, standard orbifold cohomology, earlier LG catalogues, and the established BV/BHCR relation, and (v) either demonstrates the heterotic state-space interpretation or limits the physical claims.