Referee Report

Temporal Entanglement from Twist Correlators in 2d Conformal Field Theory and Holography

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Summary and overall assessment

The manuscript proposes to define the timelike Renyi entropy of a single ordered pair of endpoints by analytically continuing the Euclidean replica-twist two-point function to a Lorentzian time-ordered correlator, equations (2.5)–(2.7). The ordering fixes an $i\varepsilon$ prescription, and the replica limit produces a complex quantity whose imaginary part changes sign under reversal of the ordering. In the vacuum on the line, equations (2.11)–(2.14) recover the familiar real logarithm and the phase $ic\pi/6$. In holographic theories the authors identify the corresponding semiclassical contribution with a complex boundary-anchored geodesic; equation (2.22) reproduces the same phase through the complex geodesic length.

The paper then develops three extensions. First, Section 2.3 constructs a backreacted Banados geometry for finite replica index, uniformizes it with the Roberts map, and analytically continues the defect locus to a complex cosmic brane. The endpoint data and profile in equations (2.52)–(2.53), together with the length in equation (2.55), are among the most interesting results of the paper. Second, Section 3 treats the vacuum on the Lorentzian cylinder. The integer k in equation (3.5) records successive light-cone crossings, giving an accumulated phase in equations (3.6)–(3.8); the global-AdS_3 geodesic endpoint in equation (3.13) reproduces this sheet structure. Third, Sections 4 and 5 study heavy states, local and global quenches, and their bulk duals. The authors identify identity-block monodromy channels with winding geodesics, advocate selection by the smallest real part of the action, and obtain a CFT/complex-geodesic match for the Vaidya quench in equations (5.14) and (5.31).

The ordered-correlator framework is useful and the paper contains results of clear interest to the two-dimensional CFT and holography communities. The vacuum and thermal continuations are internally consistent, and the finite-n construction, arbitrary-sheet cylinder analysis, and direct comparison with the Vaidya correlator are potentially publishable advances. However, a central part of the claimed saddle-selection rule is presently mathematically ill-defined: the minimization over all conical-defect monodromy sectors has no minimum for generic irrational defect parameter. In addition, the paper does not yet establish which complex saddles lie on the relevant continuation contour, and the Vaidya "optimality" test is not a complex stationarity test. These issues affect correctness rather than presentation alone.

Recommendation: Major revision. The central proposal and several calculations are promising, but the admissible saddle set, finite-n real-time interpretation, and Vaidya saddle analysis must be repaired or substantially qualified before publication.

Correctness, novelty, and significance

The elementary continuation checks are convincing. In particular, applying the stated $i\varepsilon$ prescription to the plane correlator produces the phases in equations (2.11)–(2.14), and the same branch is encoded geometrically by equation (2.22). Likewise, the cylinder result follows from continuously tracking both chiral light-cone factors: equations (3.5)–(3.8) are a useful way to organize the associated monodromies. At finite n, the phase $i\pi/n$ in equation (2.55), when interpreted as the refined-entropy contribution and integrated using equation (2.49), has the correct n dependence to reproduce the imaginary term in equation (2.12). These are valuable consistency checks.

The genuinely new content should nevertheless be separated more sharply from known ingredients. Timelike-separated twist insertions, complex-valued pseudoentropies, and the vacuum and thermal formulas already appear in the literature cited in Section 1. Complex geodesics for timelike correlators also predate the present work, and the local-quench identity-block machinery is inherited from earlier analyses. The principal advances appear instead to be the unified ordering prescription, the extension to arbitrary cylinder sheets, the explicit finite-n complex brane, the proposed boundary-channel/bulk-winding dictionary, and the use of the known Vaidya autocorrelator to discriminate between complex-geodesic and piecewise-curve prescriptions. Those advances would meet the interest threshold of a journal such as JHEP or Physical Review D if the saddle and domain problems below are resolved.

Major comments

1. **The all-channel minimization is ill-defined for a generic conical defect.**

   Equations (4.9), (4.10), and (4.14) minimize over every $m \in \mathbb{Z}$. In the conical regime, the quantity inside the logarithm contains

   $$|\cos(\alpha\Delta t) - \cos(\alpha(\Delta\phi - 2\pi m))|\,, \qquad 0 < \alpha < 1.$$

   For irrational α, the sequence $\alpha(\Delta\phi - 2\pi m)$ is dense modulo 2π. Hence the absolute difference above has infimum zero along a subsequence. The logarithm is therefore unbounded below, and the displayed minimum generally does not exist. The problem is already present in the purported ordinary entropy (4.10): at $\Delta t = 0$, the factor $1 - \cos[\alpha(\Delta\phi - 2\pi m)]$ can approach zero arbitrarily closely. It is not a peculiarity of the timelike continuation.

   The same pathology appears on the bulk side in equations (4.19)–(4.21) and in the conical-defect discussion around equations (A.33)–(A.35). Appendix A notes that rational α gives only finitely many geometrically distinct windings, but the main text treats arbitrary heavy dimensions and hence generic irrational α. Moreover, for a genuine orbifold $\alpha = 1/N$ a finite image set is natural, whereas extending the same image prescription to an arbitrary cone requires additional global input. Formally assigning an integer winding to every inverse-trigonometric branch does not by itself show that each branch is a distinct admissible geodesic with the stated endpoints and homology.

   The problem also propagates into the local-quench discussion. Applying the manuscript's channel rule to equation (4.32) replaces the factors z^α and $\bar{z}^\alpha$ by $e^{-2\pi i\alpha m}z^\alpha$ and $e^{2\pi i\alpha m}\bar{z}^\alpha$. For the Lorentzian trajectories in Section 4.3, $|z| = |\bar{z}| = 1$. With irrational α, the phases generated by m are dense, so at least one of the factors $1 - e^{-2\pi i\alpha m}z^\alpha$ and $1 - e^{2\pi i\alpha m}\bar{z}^\alpha$ can be made arbitrarily small. The real part again has infimum $-\infty$. Consequently, the statement after Figure 4 that all other channels are subleading, and the subsequent $m = 0$ or $m = 1$ choices used in deriving equations (4.43)–(4.44), do not follow from the all-integer prescription as written.

This is a direct correctness problem in one of the manuscript's central results. The authors should define the admissible set of CFT channels and bulk curves before minimizing. Possible restrictions might involve channels reached from a fixed Euclidean OPE decomposition, simple geodesics, a homology condition, or a finite orbifold image set, but any choice must be derived rather than imposed after the fact. If non-vacuum blocks, semiclassical coefficients, or contour accessibility regulate the dense sequence, that mechanism must be exhibited. Equations (4.9), (4.10), and (4.14), and the corresponding claims of a general boundary derivation of minimum-real-length selection, must then be recomputed within the corrected domain.

2. **Minimum real part is not by itself a derivation of the contributing saddle set.**

Equation (4.3) assumes vacuum Virasoro-block dominance, while equation (4.4) gives one semiclassical block. The manuscript then treats its integer monodromy images as competing saddles and selects the one with the smallest real entropy. A semiclassical solution contributes to a Lorentzian correlator only if its steepest-descent cycle intersects the analytically continued Euclidean contour, and it can carry a nontrivial coefficient and phase. These data are not determined by comparing $\mathrm{Re}\, S_m$ alone. Lorentzian continuation can cross Stokes lines, precisely where the set of contributing saddles changes.

The degeneracy discussed after equations (4.14)–(4.16) makes the gap concrete. At $\Delta\phi = 0$, the $+|m|$ and $-|m|$ sectors have equal real parts. Continuing from an infinitesimal nonzero spatial separation selects a one-sided limit, but it does not determine the correlator at the degenerate point. There the leading answer is generally the logarithm of a sum of saddle contributions, including their prefactors and phases, rather than one member selected by fiat. The same issue arises at channel-transition loci away from $\Delta\phi = 0$.

The authors should either derive the contributing monodromy sectors and their coefficients from a controlled conformal-block decomposition and its Lorentzian continuation, or explicitly restrict the claim to a leading approximation away from degeneracies and Stokes loci. In the latter case, the abstract and conclusions should no longer state that the boundary correlator universally selects the smallest-real-length geodesic. The notation in equations (4.14) and (4.16) should also introduce a well-defined $m_\star$ and a rule for ties, since the imaginary term presently retains an m outside the displayed minimization.

3. **The finite-n geometry is a formal continuation, but its status as a real-time replica saddle remains to be shown.**

Section 2.3 gives an elegant local construction. The twist stress tensor determines the Banados metric, the uniformization map produces the conical quotient, and equations (2.52)–(2.55) analytically continue the endpoints, defect profile, and length. This establishes a complex locally AdS geometry with the desired algebraic monodromy. It does not yet establish that the geometry lies on the integration cycle of the Lorentzian replica or Schwinger-Keldysh gravitational path integral.

Several checks are needed for the claim of an explicit complex cosmic-brane prescription at finite n: the global replica boundary conditions, the local monodromy and brane equation with tension $T_n = (n-1)/(4nG_N)$, the gluing or timefold associated with the ordered correlator, and the relation of the chosen complex saddle to the Euclidean integration contour. A calculation of the renormalized on-shell action would be especially useful, because the transition from the regulated expression (2.45) to the finite term (2.47) discards n-dependent regulator contributions. Although insertion-independent terms do not affect all kinematic comparisons, they do affect the normalization of Renyi quantities and should be handled by a stated counterterm prescription rather than simply removed.

The authors should compare the construction more explicitly with the real-time replica analysis cited near equation (2.55). If the intended claim is narrower—namely, that analytic continuation of the Euclidean quotient supplies a candidate complex brane whose separation-dependent area reproduces the CFT formula—that would still be worthwhile, but it should be stated at that level.

4. **The definition must include the continuation path and logarithmic branch, and its relation to temporal entanglement needs clarification.**

   Equations (2.5)–(2.7) define a legitimate ordered correlator-derived observable. Time ordering fixes how the correlator approaches a light-cone singularity, but equation (2.7) also contains an outer complex logarithm. Its value is not fixed by the final correlator alone. The accumulated integer k in equations (3.5)–(3.8) is obtained by continuously transporting the insertions from a Euclidean or spacelike base point through a specified sequence of light-cone crossings. That homotopy data should be part of the definition. Otherwise, the repeated claim that operator ordering "uniquely" fixes the imaginary part is too strong: ordering fixes the local $i\varepsilon$ side, while a base sheet and continuation path fix the global logarithmic branch.

   There is also a conceptual distinction between the quantity actually constructed and the temporal density matrices motivating Section 1. The paper does not construct a non-Hermitian reduced operator, show that the ordered twist correlator equals its replica traces, or derive an equivalence to transverse contraction in tensor networks. It defines an analytic continuation of a replica correlator. That may be a useful definition of timelike pseudoentropy, but the manuscript should either provide the missing operational/path-integral bridge or consistently distinguish "correlator-defined TEE" from temporal entanglement measures based on an auxiliary temporal Hilbert space. This distinction is important when discussing positivity, entropy inequalities, homology, and entanglement production.

5. **The claimed Vaidya optimality test is not a complex saddle-point test.**

   The matching at the physical thin shell $v_s = 0$ and the resulting length (5.27) reproduce the known boundary correlator, which is a strong and useful check. The subsequent paragraph attempts to show that this solution is optimal among crossings displaced along imaginary v_s. However, after extremizing over y_s as in equation (5.30), substitution into the segment lengths (5.22) and (5.25) gives, up to the fixed branch,

$$\mathcal{L}(v_s) = 2\log\left[\frac{F(v_s)}{\varepsilon}\right] + i\pi,$$

   with

$$F(v_s) = 2y_H \sinh\left(\frac{t_2 - v_s}{2y_H}\right) - (t_1 - v_s)\cosh\left(\frac{t_2 - v_s}{2y_H}\right).$$

   Its derivative at the proposed point is

$$F'(0) = \frac{t_1}{2y_H}\sinh\left(\frac{t_2}{2y_H}\right) \neq 0 \qquad (t_1 < 0 < t_2).$$

   Thus $v_s = 0$ is not stationary under a general complex variation. If $\mathcal{L}'(0)$ is real, then $\mathrm{Re}\,\mathcal{L}(iu)$ is automatically stationary at $u = 0$ by the Cauchy-Riemann equations: the linear variation $iu\,\mathcal{L}'(0)$ is purely imaginary. A minimum of the real part along that single imaginary line therefore does not demonstrate a complex saddle or a Lefschetz thimble. The argument should be removed or replaced by the full complex stationarity and contour analysis. The physical shell may of course fix $v_s = 0$ without such a variation; that is a different statement and is compatible with the successful match in equation (5.31).

The thick-shell computation in Appendix B also needs substantially more validation. The mass function in equations (5.32) and (B.9) has complex poles at $v = i\pi(k + 1/2)/\gamma$, so contour deformations are not automatically harmless and the thin-shell limit is nonuniform on fixed complex contours. This can be seen directly from the comparison contour (B.12). For $t_2 = -t_1$ it passes through $v(1/2) = i\eta\Delta t$, where

$$m(v(1/2)) = \frac{m_0}{2}\left[1 + i\tan(\gamma\eta\Delta t)\right].$$

There is no $\gamma \to \infty$ limit at fixed η; poles occur at $\gamma\eta\Delta t = \pi/2 + \pi k$. For the Figure 11 values $\eta = 0.05$ and $\Delta t = 1.5$, the first pole is at $\gamma \simeq 20.94$, only moderately above the displayed $\gamma = 16$. A controlled limit requires $\eta = O(1/\gamma)$ or an explicitly deformed contour with a demonstrated branch and thimble prescription. The numerical results should also report the cutoff, discretization, residuals, and convergence; demonstrate stability under allowed contour changes in equations (B.6)–(B.12); and search for competing solutions. Figure 5 establishes continuity along one numerical branch, not uniqueness of the gravitational saddle.

6. **The validity regime of the heavy-state and quench conclusions must be stated more carefully.**

   Section 4 relies on the large-c, sparse-spectrum, identity-block approximation in equations (4.3)–(4.5). Exact heavy-state correlators on a compact circle contain non-vacuum blocks and finite-volume recurrences, and an individual pure heavy state is not identical to the thermal BTZ ensemble. The language of "BTZ microstate geometry" in Sections 4.1–4.2 combines the classical BTZ exterior with a purity-motivated omission of the usual horizon/homology configuration; the domain in which this hybrid description follows from the heavy-state block should be stated explicitly. In particular, arbitrary temporal extent and large monodromy can probe precisely the regime where neglected blocks or microstate data matter.

   The same qualification applies to the imaginary-part claims. The vacuum and thermal two-point functions fix the phase exactly within their stated CFT settings, but equations (4.14)–(4.16) are leading semiclassical, channel-dependent expressions. Generic blocks can introduce additional Lorentzian monodromies. The statement that the imaginary part is independent of dynamics should therefore be limited to the examples and approximations actually analyzed.

   In particular, the α-rescaled zeros in equations (4.14)–(4.15) are singularities of an individual semiclassical identity block. They are not automatically physical light cones or singularities of the exact cylinder correlator. Near these "forbidden singularities," neglected blocks and finite-c corrections are expected to matter, so the phase jumps cannot be interpreted as exact causal data without a validity estimate. The paper should delimit a kinematic region separated from these zeros and explain what replaces the single-block answer near them.

   Similarly, equations (4.43)–(4.44) show that the real part of the local-quench quantity differs from its vacuum value while exactly one endpoint lies across the excitation light cone, and equations (5.13)–(5.14) give the analogous global-quench correlator. This is a clean causal statement. It is not yet a derivation of quasiparticle counting or of temporal entanglement production, because no temporal reduced operator or counting rule is constructed. The discussion should use the narrower interpretation unless such a bridge is supplied.

7. **The novelty claims and comparison with competing prescriptions should be made explicit and quantitative.**

   Section 1 acknowledges that timelike twist insertions were previously used in pseudoentropy, TEE, and timelike Renyi calculations, and that complex geodesics have a longer history. The manuscript would benefit from a concise statement, example by example, of what is known input and what is new here. In my view, the strongest new elements are the unified ordered-twist formulation, the arbitrary-sheet cylinder result, the candidate finite-n complex brane, and the direct Vaidya correlator/geodesic comparison.

The rejection of piecewise-curve prescriptions is currently broader than the displayed evidence. For global AdS_3 and Vaidya, the authors should place the complex-geodesic and piecewise formulas side by side, specify the same boundary data and regulators, and identify the precise term or time dependence that disagrees with the boundary correlator. This is especially important for the works discussed around equations (5.31)–(5.32), and for the composite-geodesic literature cited in the Introduction. Such a comparison would make the significance of the CFT result much clearer and avoid conflating an independent geometric observable with the replica-defined TEE.

Several directly relevant works already present in the bibliography should also be discussed where the corresponding claims are made: "Imaginary part of timelike entanglement entropy" (arXiv:2410.22684) in Section 3; "Relation between time- and spacelike entanglement entropy" (arXiv:2402.00268) near the basic continuation proposal; "Non-hermitian Density Matrices from Time-like Entanglement and Wormholes" (arXiv:2512.13800) in connection with the motivating density-matrix interpretation; and "Entanglement first law for timelike entanglement entropy and linearized Einstein's equation" (arXiv:2511.17098) before Section 6 presents a first-law analysis as a future starting point. The replica/time-contour works arXiv:2509.02775 and arXiv:2604.00108 are also directly relevant to the finite-n and de Sitter discussions.

Minor comments

1. State the kinematic domains used in equations (3.5), (4.15), and (4.16), including the representative range for $\Delta\phi$, the convention at an exact light-cone crossing, and the value assigned to the step function at zero.

2. In equations (4.14) and (4.16), write each channel as a complete complex quantity S_m and then define the selected channel. The present line break visually places the imaginary term outside the minimization and is ambiguous even away from degeneracies.

3. Section 4.4 states that the $m = 0$ geodesic is the only relevant saddle for the local-quench kinematics. This should be demonstrated by comparing its real part and admissibility with the nonzero-winding sectors, especially in view of the problem in comment 1.

4. The phrase "any path" in the complex affine-parameter plane, used in Sections 2.2 and Appendix B, should be restricted to contours in the same allowed homotopy class that avoid singularities and preserve the chosen square-root branches. The thick-shell mass function makes this restriction substantive.

5. The numerical analysis associated with Figure 5 should include error bars or residual measures. The inset at the level of 10^{-10} is not informative without the solver tolerance, precision, cutoff extrapolation, and contour-deformation tests.

6. The citation to a "work in progress" in Section 6 should not be used as evidence for a substantive general claim about nonconformal theories. The claim should be identified as preliminary or supported by a public calculation.

7. The bibliography data contain a stray duplicate set of fields for the Kanda–Takayanagi–Wei paper, and one author is duplicated under two spellings in the Guo–Jiang entry. These should be cleaned before publication.

8. The conclusion should distinguish three levels of statement: exact consequences of a specified two-point function, results in the leading large-c identity block, and conjectures about a gravitational integration contour. This would prevent several local consistency checks from being read as universal saddle theorems.

9. It would help to display explicitly the finite-n refined entropy obtained from equation (2.55) before integrating equation (2.49), including the retained and subtracted regulator terms. This would make the complex-brane result independently reproducible.

10. The manuscript is generally well organized, but Section 6 is long relative to the unresolved issues in the main derivation. Some speculative material on tensor networks and de Sitter space could be shortened in favor of a sharper discussion of saddle admissibility, Renyi normalization, and the limits of identity-block dominance.

Recommendation

I recommend major revision. The paper has a coherent central idea and contains several technically interesting results, particularly the finite-replica-index complex-brane construction and the boundary-controlled Vaidya comparison. The vacuum, thermal, and cylinder calculations provide a strong foundation. However, the unrestricted conical-channel minimum is not defined for generic heavy states, the minimum-real-part rule lacks the required contour and Stokes data, and the Vaidya optimality argument does not test complex stationarity. A revised manuscript that corrects the channel domain, establishes or carefully qualifies the relevant complex saddles, and narrows the physical claims would be a valuable contribution.