# Referee Report

*Resurgent Lambert series from Feynman and beyond*

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**Recommendation: Reject in its present form as a full research-journal article.** The manuscript could serve as a useful conference-proceedings overview after substantial technical correction, but it does not presently meet the originality, correctness, and self-containment standards of a journal such as JHEP or Physical Review D. A future journal submission would need both corrected formulae and substantial new results beyond the authors' earlier work, not merely a longer exposition.

## Summary of the paper

The paper develops a common narrative for several appearances of Lambert series in mathematical physics. Section 2 begins with Ramanujan transformations of the untwisted series $T_s(\tau)$. It then applies the level-six modular parametrization of the equal-mass three-loop banana integral $J(t)$, relating its expansion near $t = 0$ to its large-momentum expansion through a Fricke involution. The stated conclusion is that the same modular relation connects the coefficient $7\zeta(3)$ at the origin with the constant $16\zeta(3)$ in the large-$t$ asymptotic expansion.

Section 3 interprets the finite Laurent polynomial in Ramanujan's odd-weight transformation as the perturbative part of a transseries whose exponentially small term survives after the perturbative series terminates. This is the "Cheshire cat" mechanism. The two-loop sunrise integral supplies a character-twisted example: although its Lambert index is even, the relevant character $\chi_{6,5}$ is odd, and the authors argue that the perturbative modularity gap again terminates. Equations (27)–(29) introduce ordinary and dual character-twisted Lambert series and give a quadratic-character transformation in terms of Bernoulli polynomials.

Section 4 introduces the two-character series $\Xi_{s_1,s_2}(\chi_{r_1}, \chi_{r_2}; \tau)$. Equation (31) is meant to display the transformed, exponentially suppressed sector that completes the resummed perturbative answer. Finally, Section 5 rewrites a fermionic spectral trace for local $\mathbb{P}^2$ in terms of a Fricke-paired vector of Lambert series, and Section 5.1 sketches the corresponding character structure for local $\mathbb{P}^{m,n}$. The $N = 5$ examples contain two odd quartic characters and one even quadratic character, so that ordinarily resurgent and Cheshire-cat sectors occur together.

This is an attractive unifying theme. The connection among modular Feynman periods, character parity, and strong–weak transseries is potentially useful to both the Feynman-integral and topological-string communities. Several elementary consistency checks are also successful. For example, equation (23) specializes correctly to the transformations in equations (7)–(9), the reductions in equations (32)–(33) follow directly by exchanging the two summation variables, and the logarithmic identity in equation (36) is consistent with the products in equations (34)–(35) once branches are fixed. Unfortunately, three central displayed formulae have definite normalization or sign errors, and most of the claimed physical applications are presented too briefly to establish their scope.

## Assessment of novelty and significance

The manuscript reads as a concise conference review, not as an original research article. Equations (1)–(23) assemble Ramanujan identities and established banana-integral results from the cited literature. The sunrise parametrization in equations (24)–(26) is likewise attributed to earlier work. Section 4 explicitly says that the two-character analysis was obtained in the authors' earlier paper, Ref. [10] (arXiv:2507.21352). More importantly, that same prior paper already contains the local-$\mathbb{P}^{m,n}$ analysis and the $N = 5$ coexistence of odd and even character sectors in its Section 6.2. Thus the most conspicuous candidate for a new result in the present article is also a compressed account of previously public work.

There is value in a well-written proceedings synthesis, especially because the parity criterion gives a simple conceptual explanation of when an exponentially small sector survives a terminating perturbative expansion. However, the manuscript does not derive a new Feynman integral, a new spectral trace, a new modular transformation, or a new classification theorem. Nor does it explain a new physical consequence of the $N = 5$ mixture. Under research-journal standards, this is a decisive novelty problem independent of the technical errors listed below.

## Major comments

1. **The paper must state precisely what is new and choose a publication scope consistent with that content.**

   Section 1 should contain an explicit provenance map. In particular, the authors should say whether equations (21)–(22) are new derivations or a reformulation of known banana-integral results; identify equations (30)–(33) as a summary of Ref. [10]; and acknowledge that the local-$\mathbb{P}^{m,n}$ character decomposition and the mixed $N = 5$ example already appear in Section 6.2 of that same reference. If this is intended only as an LL2026 proceedings contribution, that review character is reasonable, but the technical exposition still needs correction. If it is intended as a standalone journal article, it needs a genuinely new result and a quantitative demonstration of its physical or mathematical significance.

2. **The Taylor-series normalization before equation (14) is incompatible with both the recurrence and the Picard–Fuchs equation (15).**

   The text defines $J(t) = \sum_{n\geq 0} C_n t^n$ and then gives equation (14) together with $C_0 = 7\zeta(3)$ and $C_1 = C_0 - 6$. This cannot be correct with the stated definition. Taking the constant term of equation (15) gives

   $$64C_1 - 4C_0 + 24 = 0,$$

   and therefore

   $$C_1 = \frac{C_0 - 6}{16}.$$

   More generally, if $c_n$ denotes the coefficient of $t^n$, equation (15) yields

   $$c_n = \frac{1}{16}\left(1 - \frac{1}{2n}\right)\left(5 - \frac{5}{n} + \frac{2}{n^2}\right)c_{n-1} - \frac{1}{64}\left(1 - \frac{1}{n}\right)^3 c_{n-2}.$$

   The printed recurrence and the printed initial value are instead consistent with

   $$J(t) = \sum_{n\geq 0} C_n \left(\frac{t}{16}\right)^n.$$

   This is not merely a stylistic normalization issue: the series as printed fails its own differential equation already at order $t^0$. The authors should correct the expansion and carry the convention consistently through the discussion of the singular points and convergence domain.

3. **Equation (29) has the wrong sign in every stated $D < 0$, even-$s$ case.**

   The problem can be seen without any delicate asymptotics. Take the primitive quadratic character $\chi_{-4}$, set $s = 2$, and choose $\tau = 0.7\,i$. With the definitions in equations (27)–(28), the left-hand side of equation (29) as printed evaluates to $1.7308101565\,i$, whereas its Bernoulli-polynomial right-hand side evaluates to $2.0326685255\,i$. Replacing the transformed factor $2(-\tau)^{s-1}\widetilde{\mathcal{L}}_s$ by $2\tau^{s-1}\widetilde{\mathcal{L}}_s$ gives $2.0326685255\,i$ and restores the identity. For $D > 0$ the paper takes $s$ odd, so the two powers happen to coincide; this is why the sign error is hidden in the positive-discriminant cases.

   The authors should correct the formula, state the branch convention for $\sqrt{D}$ and powers of $\tau$, and check the corrected identity in at least one positive- and one negative-discriminant example. Since equation (29) supplies the explicit modularity gap on which the Cheshire-cat argument rests, this error affects a central claim rather than an incidental example.

4. **Equation (31) omits an essential character- and conductor-dependent prefactor and is internally inconsistent with equation (37).**

   The transformed sum in equation (31) is missing the factor

   $$r_1^{\frac{1}{2}-s_1} r_2^{\frac{1}{2}-s_2} \epsilon(\chi_{r_1})\epsilon(\chi_{r_2}) i^{\kappa_1+\kappa_2},$$

   where $\kappa_i$ is the character parity and $\epsilon(\chi_{r_i})$ is the normalized Gauss/root number. This factor appears in the detailed result quoted as equation (3.46) of Ref. [10]. Its absence can also be diagnosed entirely within the present manuscript. Specialize equation (31) to $(s_1, s_2) = (1, 0)$, $(r_1, r_2) = (3, 1)$, and $\chi_{r_1} = \chi_{3,2}$. Only the $n = 0$ term survives, and the printed equation predicts the transformed contribution with coefficient one. The first row of equation (37), however, assigns the same transformed dual Lambert series the coefficient $\pm i/\sqrt{3}$. The two central formulae therefore contradict each other.

   Equation (31) should be replaced by a complete statement, including the missing prefactor, the resummed perturbative term, the lateral prescription, and all hypotheses. The paper should define $\epsilon(\chi)$, $\kappa_i$, $\bar{\chi}$, and the Pochhammer symbol $(s)_n$. It should also say explicitly that the displayed infinite $n$-series is a formal transseries requiring suitable truncation in generic cases, rather than presenting it as an ordinary convergent equality.

5. **The banana-integral argument omits the steps needed to support its main small/large momentum claim.**

   Equation (18) is third order in $q\,d/dq$. Integrating it to equation (19) permits three homogeneous integration terms. Checking only equation (20) fixes one piece of boundary data and does not explain why all remaining terms vanish or are absorbed into the chosen period. Likewise, the passage from the Fricke relation (21) to the asymptotic formula (22) requires the transformations of $t(\tau)$ and $\psi(\tau)$, the choice of sheet for $t$, and a branch of $\log(-t)$. None is displayed.

   The authors should give the boundary conditions that select equation (19), derive the Fricke action on both $t$ and $\psi$, and state the complex sector in which the remainder estimate in equation (22) holds. They should then show explicitly how $J(0) = 7\zeta(3)$ produces the different coefficient $16\zeta(3)$ at infinity. This bridge is the principal physical message of Section 2, and it merits more than an asserted final line.

6. **The sunrise example does not establish the claimed Cheshire-cat behavior of the physical integral.**

   After equation (25), the paper says only that integration produces equation (26) “inter alia.” The complete expression for $I(w^2)/f$, its other Lambert-series terms, and its integration constants are absent.

Consequently one cannot infer from the displayed equations that the single series in equation (26) controls the expansion of the full sunrise integral at each cusp.

There is also an important arithmetic distinction. The character $\chi_{6,5}$ is a character modulo six induced from the primitive character $\chi_{3,2}$ of conductor three. Equation (29), by contrast, is stated only for primitive quadratic characters whose conductor is $|D|$. The authors should give the full integrated sunrise formula and either display the imprimitive transformation or show the finite level projection that reduces $\chi_{6,5}$ to $\chi_{3,2}$. The actual terminating Laurent part and the exponentially suppressed term should then be written explicitly at the relevant cusp.

7. **The resurgent statements need definitions and a checkable ambiguity-cancellation argument.**

   Equation (37) introduces $\mathcal{S}_{\mp}[\mathcal{L}^{\mathrm{Pert}}]$, but the perturbative coefficients, Borel transforms, Stokes ray, lateral-contour convention, and correlation between the $\mp$ and $\pm$ signs are not defined. Equation (31) is even less complete: it displays only a transformed sum without stating the equality of which it is a part. A reader cannot verify from the paper that the discontinuity of the perturbative sector is canceled by the advertised nonperturbative sector.

   At minimum, the authors should work through one component of equation (37): derive its large-order coefficients, locate the leading Borel singularity, compute the lateral discontinuity, and show that the transformed Lambert series cancels it with the stated matrix coefficient. A numerical comparison at one value of $\tau = iy$ among the exact product, optimally truncated lateral sums, and the completed transseries would fix all signs and normalizations. This would also help distinguish a formal asymptotic relation from an exact identity on the upper half-plane.

8. **Section 5.1 overstates the general local-$\mathbb{P}^{m,n}$ conclusion and confuses modulus with conductor.**

   The statement that the relevant characters have "conductor $N = m + n + 1$" is false in general. The decomposition naturally uses characters modulo $N$; for composite $N$, some of these may be imprimitive and have smaller conductor. The manuscript's own $\chi_{6,5}$ example illustrates the difference. This matters because the primitive formula and its Fricke normalization do not apply unchanged to an induced character.

   Moreover, the examples $N = 3, 4, 5$ do not prove a general local-$\mathbb{P}^{m,n}$ statement. For $N = 5$, the paper lists the characters but displays only the even quadratic component in equation (39); it does not give the spectral-trace product, the coefficients with which all three non-principal characters enter, or the dependence on the separate choices $(m, n) = (3, 1)$ and $(2, 2)$. The authors should provide the general product or trace formula, perform its complete character decomposition, and explain how primitive and imprimitive sectors are treated. They should then state the physical payoff: for example, whether the decomposition yields new spectral data, a more efficient numerical strong–weak continuation, or a genuine classification principle for toric geometries.

## Minor comments

1. State the range $k \geq 1$ in equation (23) and the domain of $\tau$ wherever a Fricke transformation is used. The conditions $\operatorname{Im}\tau > 0$ and the relevant limiting sector should not be left implicit.

2. Equation (26) writes $q$ as the second argument of $\mathcal{L}_2$, whereas equations (27)–(29) use $\tau$. Either define two notations explicitly or use one convention throughout.

3. In equation (29), state that the infinite sum is $L(s, \chi_D)$, define the fundamental discriminant convention, and specify all allowed positive integers $s$. The branches of $\sqrt{D}$, $(-\tau)^{s-1}$, and the transformed argument also need to be fixed.

4. Equation (35) should define the standard two-argument Pochhammer symbol $(a;q)_\infty$ first. The specialized notation $(xq^\alpha; q)_\infty$ makes the products with bases $q^3$ and $\widetilde{q}$ unnecessarily hard to parse.

5. Equation (36) depends on logarithm and square-root branches. The identity is consistent on the principal branches for positive $y = -i\tau$, but the paper should state the domain and the possible additive multiples of $2\pi i$.

6. In equation (39), the final Lambert series uses a comma between the character and transformed argument, whereas the notation everywhere else uses a semicolon.

7. The sentence following equation (17) should read "the square of an elliptic integral" and should be joined grammatically to the statement identifying $\psi$ as a homogeneous solution. Several sentences in Sections 2 and 5 would benefit from a careful proofreading pass.

8. The paper should cite the precise original or prior equation for the Fricke identity (21) and the asymptotic formula (22), or explicitly mark and derive them as results of the present note.

## Conclusion

The manuscript has a clear and potentially valuable organizing idea: character parity distinguishes factorially divergent modularity gaps from terminating gaps that retain nonperturbative data. In its present form, however, the note neither provides new journal-level results nor states the known ones reliably. The Taylor normalization, the negative-discriminant sign in equation (29), and the missing prefactor in equation (31) are concrete correctness failures. The Feynman-integral and topological-string applications are too abbreviated to repair those failures or to demonstrate the claimed generality. I therefore cannot recommend publication as a full research article. A corrected version could be useful as conference proceedings, while a future journal submission would require a substantially new and self-contained contribution.