Referee Report

on "Non-perturbative saturation of Krylov complexity, and its implications in quantum gravity"
arXiv:2607.14220

Summary of the paper

The manuscript studies state Krylov complexity in a finite-dimensional quantum system whose energy spectrum is discrete and bounded. Starting from the equal-amplitude seed in equation (2.4), the Lanczos construction produces orthonormal polynomials on the set of energy levels and a position operator X on the resulting Krylov chain. The motivating question is whether this basis, although defined using the full spectrum, can nevertheless reproduce a saturation scale controlled by the much smaller number of states near a chosen low energy. This question is relevant to proposals identifying the Krylov index with wormhole length in double-scaled SYK (DSSYK) and its triple-scaled JT-gravity regime.

Section 2 reviews the Krylov construction and gives a heuristic connection between the asymptotic Lanczos coefficients and the growth law for $\langle X\rangle$. The authors emphasize that a bounded spectrum gives asymptotically constant Lanczos coefficients and hence permits linear rather than faster Krylov growth. Equation (2.17) then expresses the infinite-time averaged complexity in terms of the energy-basis weights of the Krylov polynomials.

The main mathematical result is Theorem 1 in Section 3. For an energy level z_k in a window of d_{edge} levels near a spectral edge, the theorem is intended to bound the index q_k by which the partial Christoffel–Darboux kernel

$$\sum_{n=0}^{q_k} |p_n(z_k)|^2$$

is close to its complete value. Appendix A constructs a trial polynomial with exact zeros on the other levels in the edge window and a normalized Chebyshev factor suppressing its values on the rest of the spectrum. The resulting estimate depends on the edge size, the width ratio η_k, and the logarithmic potential $\nu_{k,\mathrm{edge}}$. Appendix B argues that the latter admits a continuum expression under spectral-rigidity assumptions. Section 3.2 supplements the bound with a Szegő-asymptotic picture and one CUE example illustrating the withdrawal of high-order polynomials from the edge.

Section 4 applies this construction to the DSSYK/JT correspondence. With $x = \lambda n$ and $z - z_0 = \lambda E$, the authors estimate an upper localization scale of order the number of JT states below a nearby edge cutoff, multiplied by a factor logarithmic in $1/\lambda$. They contrast the fixed, infinite-temperature reference basis with Gibbs-seed and microcanonical-seed constructions, and with an "ergodic" length operator discussed in earlier work. The principal physical claim is that this fixed Krylov basis is consistent with wormhole-length saturation near the Heisenberg scale for any low-energy initial state.

Evaluation

There is a useful idea in this paper. The combination of the Christoffel variational principle with an edge-zero polynomial and a Chebyshev residual factor gives a simple, nonasymptotic way to quantify how a discrete orthogonal-polynomial basis withdraws from a sparsely populated spectral edge. This construction is transparent, and its application to the fixed $\beta = 0$ DSSYK Krylov basis addresses a genuine question left by the semiclassical Krylov/length correspondence. If made correct and properly delimited, the result should be of interest both to researchers studying Krylov dynamics and to the JT-gravity community.

The present version is not ready for publication, however. Most importantly, Theorem 1 is false as stated. Appendix A obtains an exact sufficient Chebyshev-degree condition in equation (A.13), but then replaces it by lower bounds with the wrong logical direction. The error is not merely formal: a four-level spectrum furnishes an explicit counterexample to equations (3.6)–(3.8). The exact condition in equation (A.13) appears to preserve the intended leading scaling, so this problem may be repairable, but the theorem and every downstream estimate must be rewritten and checked.

There is also a substantial gap between the mathematical quantity bounded in Section 3 and the physical conclusion advertised in the abstract. The theorem controls a pointwise cumulative weight for an individual energy eigenvector. It does not establish the time at which complexity reaches a plateau, a plateau for a coherent low-energy state, or even an $O(q_k)$ bound on the first moment of X without a dimension-dependent choice of the resolution parameter. The manuscript repeatedly promotes this kinematic localization statement to a proof that complexity "stops growing" in every low-energy state. That promotion requires additional operator and dynamical arguments.

The novelty is therefore potentially significant but narrower than presently claimed: the work offers an explicit finite-set localization bound for energy components in the full-spectrum Krylov basis and a conditional consistency check for the DSSYK/JT proposal. It does not yet establish nonperturbative Krylov or wormhole dynamics. Several normalization, energy-origin, and scaling errors also have to be corrected before the quantitative gravity comparison can be assessed.

Recommendation

I recommend **major revision**. A publishable revision should replace the incorrect theorem by a valid sufficient bound, propagate that correction through the DSSYK estimates, prove an appropriate low-energy-subspace or first-moment corollary, and sharply distinguish static Krylov-index localization from dynamical saturation. The quantum-gravity interpretation should then be presented with explicit orders of limits and with a more balanced comparison to the finite-temperature, microcanonical, and nonperturbative-length constructions already cited.

Major comments

1. **Theorem 1 is false because the last step of the proof reverses the required logic.**

   The trial polynomial argument through equation (A.13) is straightforward. It gives the sufficient condition
   $$r \geq \frac{\operatorname{arccosh}\Big[(\sqrt{d}/\epsilon)\exp(d_{\mathrm{edge}}\nu_{k,\mathrm{edge}})\Big]}{\operatorname{arccosh}(1+2\eta_k)}$$
   for the Chebyshev degree. In passing from this expression to equation (A.16), however, the manuscript uses a lower bound on the numerator $\operatorname{arccosh} Z$ and a lower bound on $1/\operatorname{arccosh}(1+2\eta_k)$. Their product is therefore a lower bound on the exact required degree. Requiring r to exceed that smaller number is not sufficient to satisfy equation (A.13). Nevertheless, equation (A.16), the last sentence of Appendix A, and equation (3.8) all treat it as a sufficient condition.

   There is a small exact counterexample. Take
   $$d = 4, \qquad (z_0, z_1, z_2, z_3) = (0, 1, 2, 3), \qquad d_{\mathrm{edge}} = 2, \quad k = 1, \quad \epsilon = \tfrac{1}{2}, \quad \gamma = 1.$$

Then $\eta_k = 1$, $\nu_{k,\text{edge}} = \frac{1}{2}\log 3$, and the condition in equation (3.6) holds. Equation (3.8) gives $q_k \geq 1.5146\ldots$, so the integer $q_k = 2$ qualifies. The first three normalized discrete polynomials have, at z_1,

$$|p_0(z_1)|^2 = \frac{1}{4}, \qquad |p_1(z_1)|^2 = \frac{1}{20}, \qquad |p_2(z_1)|^2 = \frac{1}{4}.$$

Consequently

$$\sum_{n=0}^{2} |p_n(z_1)|^2 = \frac{11}{20} = 0.55 < \frac{1}{1+\epsilon^2} = 0.8,$$

in direct contradiction with equation (3.7). By contrast, equation (A.13) requires

$$r \geq \frac{\operatorname{arccosh} 12}{\operatorname{arccosh} 3} = 1.802\ldots,$$

so integer $r = 2$ and $q = d_{\text{edge}} - 1 + r = 3$, which gives exact completeness.

The cleanest repair seems to be to state the theorem using equation (A.13) itself, with a ceiling for the integer degree, rather than forcing an elementary but invalid simplification. Alternatively, the authors need an *upper* bound on the right-hand side of equation (A.13). After making this change, equations (3.8)–(3.10) and all of Section 4 must be rederived. The leading small-η_k behavior may well survive, but that fact does not validate the present theorem.

2. **Pointwise polynomial localization is not a proof of dynamical complexity saturation.**

   Equation (3.7) implies, for one energy eigenstate and a chosen common cutoff q,

$$\tau_k(q) := \sum_{n>q} |p_n(z_k)|^2 \leq \delta, \qquad \delta = \frac{\epsilon^2}{1+\epsilon^2}.$$

   This is a useful statement about one column of the unitary change of basis. An energy eigenstate itself has no time-dependent Krylov growth. For a general state,

$$\langle K_n|\psi(t)\rangle = \sum_k c_k\, \mathrm{e}^{-\mathrm{i}z_k t} p_n(z_k),$$

   and the interference terms are not controlled by the separate diagonal estimates $\tau_k(q)$.

   Even for the infinite-time average in equation (2.17), the theorem controls a tail probability, while X weights that tail by indices as large as $d-1$. If the same q and δ apply throughout the energy support, the direct consequence is only

$$\overline{\langle X\rangle} \leq q + (d-1)\delta.$$

   Thus an $O(q)$ first-moment bound requires $\delta \lesssim q/d$, not merely a fixed small ϵ. This dimension-dependent choice must be inserted back into the theorem, including the $\log(d/\epsilon^2)$ term that is later dropped.

   For a coherent state supported on M energy levels, Cauchy–Schwarz gives at best a crude uniform estimate

$$\|(1-P_q)|\psi(t)\rangle\|^2 \leq M\delta, \qquad P_q = \sum_{n=0}^{q} |K_n\rangle\langle K_n|,$$

   and hence $\langle X(t)\rangle \leq q + (d-1)M\delta$. A state-independent statement should instead control the operator norm of $(1-P_q)P_{\mathcal{E}}$, or directly the compressed operator $P_{\mathcal{E}} X P_{\mathcal{E}}$, for a precisely defined low-energy projector $P_{\mathcal{E}}$. The logarithmic cost of the stronger choice of ϵ might be harmless in the intended hierarchy, but it must be shown.

Finally, no result in Section 3 locates a saturation time, establishes an approach to a plateau, or excludes large recurrences. Early linear growth plus a kinematic upper support scale does not by itself prove a plateau at the quotient of those two quantities. The authors should either provide the missing subspace, first-moment, and dynamical bounds, or consistently reformulate the central claim as an upper localization scale for individual energy components. The abstract, the last paragraph of Section 1, and the statements surrounding equations (4.8)–(4.15) should define whether "saturation" means a quantile of X, its expectation, a diagonal-ensemble value, a typical late-time value, or an all-time ceiling.

3. **The extrapolation from the semiclassical DSSYK/JT correspondence to exponentially large Krylov index is an additional physical assumption.**

   Equations (4.3)–(4.4) summarize the triple-scaled relation $x = \lambda n$, $z - z_0 = \lambda E$. The cited Krylov/length correspondence is established in the continuum, small-x regime in which the Krylov chain reproduces equation (4.1). Section 4 then applies the same identification at indices of order the finite-dimensional edge Hilbert space, precisely where discreteness and nonperturbative corrections dominate. Theorem 1, even when corrected, is a statement about the chosen Krylov basis; it does not prove that $x = \lambda n$ remains the physical JT length operator at those indices.

   This distinction matters for the comparison with the nonperturbative bulk Hilbert-space and reconstruction literature cited in the manuscript. The finite-temperature Krylov construction and the ergodic/reconstruction construction make different assumptions about how a finite chain should complete the semiclassical length operator. The present result is valuable as a consistency constraint on the fixed $\beta = 0$ basis, but it does not select that completion or show equality to a nonperturbative gravitational observable.

   The order of limits must also be stated. The analysis mixes finite N, $S_0 \to \infty$, $\lambda \to 0$, $d_{\text{edge}} \to \infty$, and exponentially late time. The correction $\alpha \lambda p^4$ in equation (4.16) is small at fixed time and momentum, but this observation does not make its effect uniformly small at times of order e^{S}; a small Hamiltonian correction can accumulate a large phase over such times. The gravity conclusion should therefore be conditional unless the authors establish uniform control or compare the finite-dimensional x_K to an independently defined nonperturbative JT length operator.

4. **Several errors and unstated approximations affect the claimed saturation scale.**

   First, equation (3.10) is not invariant under shifting the zero of energy. The edge choice should be written as

   $$z_{d_{\text{edge}}} - z_0 \approx (C+1)(z_k - z_0),$$

   and the square root should contain $C(z_k - z_0)$, not Cz_k. As printed, the expression is not even real for a lower edge with $z_k < 0$. Section 4 implicitly makes the corrected replacement when it uses $z_k - z_0 = \lambda E_k$, so the notation and derivation should be repaired at their origin.

   Second, equations (4.8) and (4.11) imply that the bracket in equation (4.12) contains

   $$\sqrt{\frac{J}{CE}}\left[\log\frac{4J}{\lambda^2} - L_{\text{edge}}(E)\right] + \lambda.$$

   The printed equation leaves $L_{\text{edge}}(E)$ outside the square-root prefactor. The same correction must be propagated to equation (4.13).

   Third, equation (3.10) drops the $\log(d/\epsilon^2)/(2d_{\text{edge}})$ and γ-dependent terms from the purported theorem without stating the hierarchy that makes them negligible. This is especially important after ϵ is chosen small enough to control a first moment or a whole energy subspace, as discussed above.

The introduction's summary $e^{S(E)} \log E_{\max}$ also suppresses the factor $\sqrt{E}$, the use of $\mathcal{N}((C+1)E)$ rather than $\mathcal{N}(E)$, the cutoff C, and the resolution dependence.

Finally, equation (4.12) is an upper bound on a localization threshold, while equation (4.14) concerns the number of vectors needed for simultaneous completeness throughout an entire edge subspace. It is not a lower bound on the localization index of an individual energy component or on a dynamical plateau for an arbitrary initial state. The claim that the actual value is "very close" to the ideal result in equation (4.5) is therefore stronger than the two bounds establish. The authors should optimize over C, state the energy regime in which equation (4.13) is used ($E \gg J$), and present the result as an interval for a precisely defined quantity.

5. **The normalization in the intuitive orthogonal-polynomial argument is exponentially inconsistent.**

Section 2 correctly introduces the discrete density $\rho(z) = e^{S_0}\rho_0(z)$, and equation (2.13) normalizes the continuum polynomials using the full ρ. Section 3.2 instead says that the continuous polynomials in equation (3.12) are normalized with respect to ρ_0, and then identifies them directly with the discrete overlaps in equation (3.13). If P_n is normalized against ρ_0, the corresponding discrete amplitude is

$$\langle z_m | K_n \rangle \approx e^{-S_0/2} P_n(z_m),$$

up to the Jacobian conventions. Equivalently, equation (3.12) should use the full density ρ.

The missing factor propagates to equations (3.14)–(3.15). The crowding index should scale like the actual local number of levels,

$$N_* \sim \rho(z_m)\sqrt{1 - z_m^2/\Lambda^2} = e^{S_0}\rho_0(z_m)\sqrt{1 - z_m^2/\Lambda^2},$$

not the $O(1)$ expression printed in equation (3.15). As written, the intuitive argument misses precisely the Heisenberg-sized factor that motivates the paper. This section should be normalized consistently and its domain of validity stated: the quoted Szegő asymptotic is not generally uniform at the endpoints where edge scaling is central.

6. **The spectral-statistics assumptions and their applicability to DSSYK need a precise status.**

Theorem 1 is deterministic once the levels are specified, whereas the replacement of $\nu_{k,\text{edge}}$ by equation (3.9) is conditional on the assumptions in Appendix B. Those assumptions include a pointwise matching of ordered eigenvalues to a reference lattice, a polynomial lower bound on the minimum gap, a rigidity estimate for the number of points in shrinking intervals, and a particular joint thermodynamic limit. The manuscript should not describe the DSSYK continuum expression as proved unless these hypotheses are established for the relevant DSSYK ensemble and limit. At present Appendix B proves a conditional comparison and then asserts that the conditions are expected.

The citations for CUE should be tied to precise statements, particularly the claimed maximum pointwise displacement $\widetilde{\Delta}_3 = O(\log d/d)$, which is not the same object as the traditional Dyson–Mehta mean-square rigidity statistic. The paper should also distinguish ensemble expectation, convergence in probability, and almost-sure self-averaging.

There is a related finite-system issue. The theorem assumes ordered, distinct levels, while DSSYK has fermion-parity sectors and can have symmetry-class-dependent degeneracies. The application should be made within a specified irreducible symmetry sector, with the corresponding dimension and preferred phase convention for the equal-amplitude seed, or the theorem should be generalized to degeneracies. A finite-N DSSYK test of the corrected bound, including the measured $\nu_{k,\text{edge}}$ and the actual partial Christoffel kernel, would materially strengthen the quantum-gravity claim. The single CUE plot demonstrates the geometric effect but does not test the model to which Section 4 is devoted.

7. **The relation to existing complexity and nonperturbative-length proposals should be recast more carefully.**

   The mathematical novelty should be separated from the already known crowding phenomenon in discrete orthogonal-polynomial theory. Baik–Kriecherbauer–McLaughlin–Miller provide asymptotic saturation results; Simon's Christoffel principle and the residual-polynomial results supply the standard tools. The new contribution appears to be a particularly elementary finite-set bound with explicit dependence on the edge geometry, and its application to the full-spectrum $\beta = 0$ Krylov basis. Stating this precisely would make the paper's contribution clearer.

   The comparison to the Gibbs-seed construction of Balasubramanian–Magan–Nandi–Wu and the microcanonical construction of Kar–Lamprou–Rozali–Sully is currently too dismissive. The argument around equations (4.6)–(4.7) compares the zero variance of a seed-adapted operator x_β, which is true by definition, with a heuristic uncertainty estimate for a different operator x_{sc}. This shows that the two operators are not identical in distribution on that state; it does not by itself establish a "dangerous incompatibility" with a proposal that may match expectation-value dynamics or a coarse-grained observable. The inference $\Delta p \sim \Delta E/\sqrt{E}$ also requires control of the Liouville potential and correlations. If operator equality is the intended criterion, it should be stated and Δx_{sc} should be computed from equation (4.7).

   Likewise, saying that a plateau of the form $\exp[O(S)]$ differs exponentially from $O(\exp S)$ is not meaningful without specifying the coefficient hidden in the $O(S)$. The authors should quote the actual scaling result of the cited work and compare the definitions of basis, observable, state, mean versus typical behavior, and plateau time on equal footing. The nonperturbative JT path-integral and reconstruction results cited in Section 4 should also be compared by more than their nominal plateau size. Such a discussion would support the appropriate conclusion: the corrected localization bound provides a complementary, state-component-wise consistency check, not yet a replacement for those dynamical constructions.

Minor comments

1. Equation (2.12) has an incorrect displayed leading coefficient. For $\beta = 1$ and constant b_n, the exact acceleration contains
$$2\sum_m |\psi_m|^2(a_m^2 - a_{m-1}^2) \sim 4\alpha C^2 \sum_m |\psi_m|^2 m^{2\alpha-1},$$
   whereas equation (2.12) gives $2\alpha C^2$ times this power. For general β, the first term also carries a factor of β. The scaling conclusion may survive, but the formula and the conditions under which the gradient expansion is positive should be corrected.

2. The hypotheses of Theorem 1 should explicitly state that $z_0 < z_1 < \cdots < z_{d-1}$, that $d_{\mathrm{edge}} \leq d - 2$, and that q_k is an integer not exceeding $d - 1$. The endpoint $k = 0$ is excluded despite the repeated phrase "every low-energy state"; either cover it or state the restriction. Ceiling functions are needed in both the theorem and Appendix A.

3. Equation (A.1) obeys $\mathcal{K}_m(x, x) \leq 1$ at support nodes $x = z_k$, not for arbitrary real x. The equality statement after equation (A.2) describes the degree-q minimizer and should not be called a degree-m

polynomial when $m < q$. The ansatz in equation (A.6) has degree $d_{\rm edge} - 1 + r$, not $d_{\rm edge} + r$; the count used in the final sentence is the correct one.

4. The sentence before equation (3.10) has the behavior of η_k reversed. Bringing $z_{d_{\rm edge}}$ close to z_k makes η_k small, and it is the resulting $1/\sqrt{\eta_k}$ factor that enlarges the estimate.

5. Figure 1's caption says that the plotted quantity is $|\langle z_k|K_n\rangle|^2 = |p_n(z_k)|^2$, while the color bar is labeled $|p_n(z_k)|^{1/2}$. The caption should state whether a fourth-root color transform was deliberately used. The figure should also report the random seed or ensemble averaging and display a quantitative slice against the corrected bound.

6. In equation (2.10), the upper endpoint should be $d - 2$, or the conventions $a_{d-1} = 0$ and $\psi_d = 0$ should be stated. Several nearby typographical errors are easy to correct, including "coeffcients," "mostly commonly," and "overly sample."

7. A concluding section would help separate three levels of the argument: the deterministic finite-spectrum theorem, the conditional continuum/DSSYK estimate, and the conjectural identification with nonperturbative wormhole length. This distinction is more important than further qualitative use of the word "saturation."